Nonlinear modes, normal forms and invariant manifolds for vibrations of nonlinear musical instruments

Unfold Mechanics for Sound and Music – IRCAM Sep. 11-12th. 2014

Olivier THOMAS

Arts et Métiers ParisTech,

Laboratoire des Sciences de l'Information et des Systèmes (LSIS), UMR CNRS 7296

Lille, France

olivier.thomas@ensam.eu

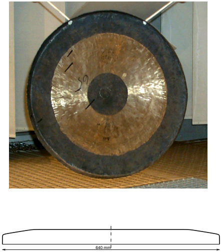
Coll.: C. Touzé (UME), Cl. Lamarque (ENTPE)

Gongs and cymbals

▷ **Thin shells...** (thickness $\simeq$ 1 mm, diameter: 20 cm to 1 m)

Cymbals

Vietnamese gong

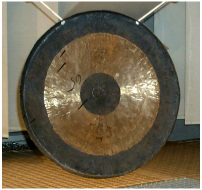


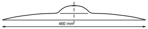
Chinese tam–tam

Gongs and cymbals

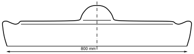
▷ **Thin shells. . .** (thickness $\simeq$ 1 mm, diameter: 20 cm to 1 m)



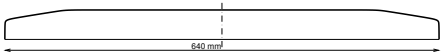




Cymbals



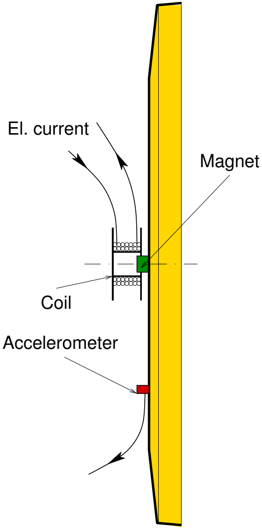
Vietnamese gong

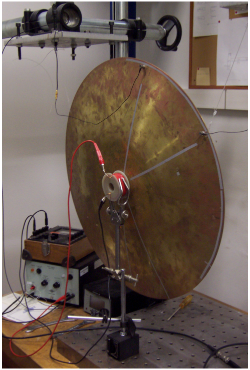


Chinese tam–tam

▷ **. . . in large amplitude vibrations**

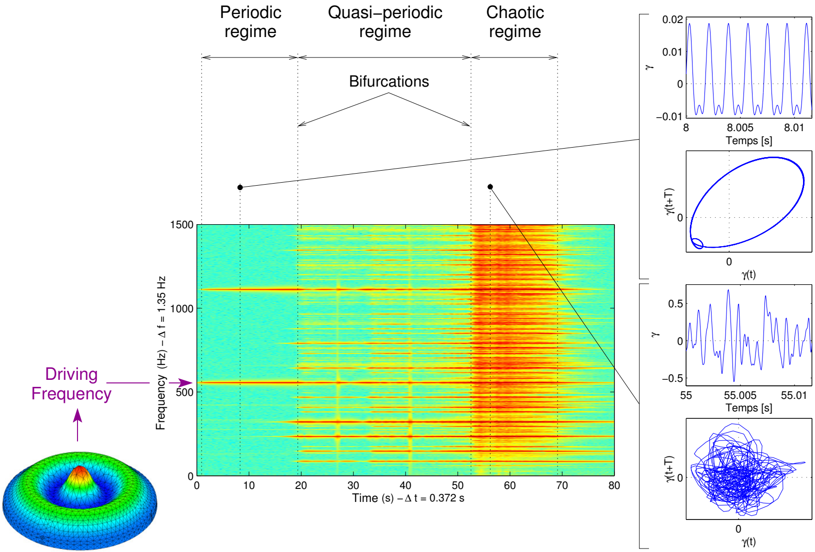

Experimental analysis: forced vibration





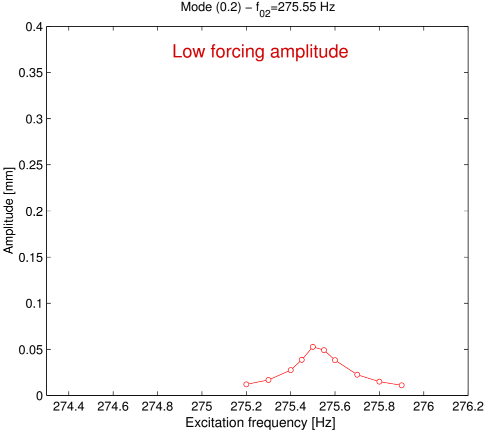
- sine forcing at center,
- constant excitation frequency close to one natural frequency
- increasing amplitude
- accelerometer measurement

Non-linear forced response



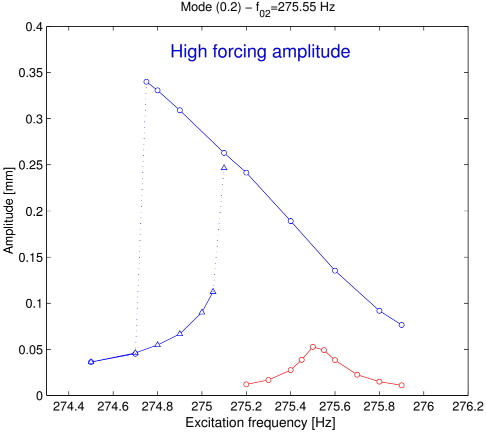
[Chaigne *et al.*, Acoust. Sci. & Tech., 2005], [Touzé, Thomas, Amabili, IJNLM 2010]

Periodic regime analysis



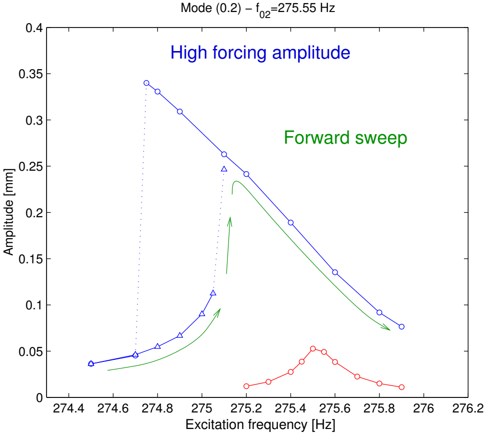
- Resonance curve: constant forcing amplitude, frequency sweep

Periodic regime analysis



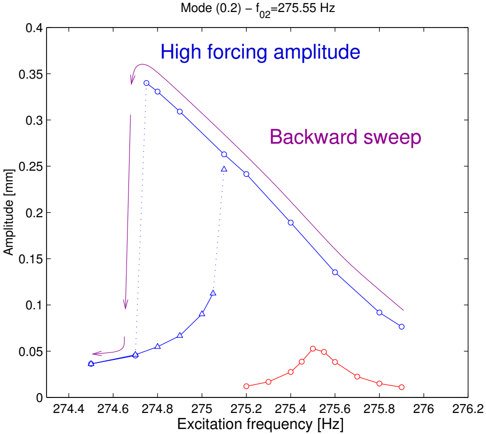
- Resonance curve: constant forcing amplitude, frequency sweep

Periodic regime analysis



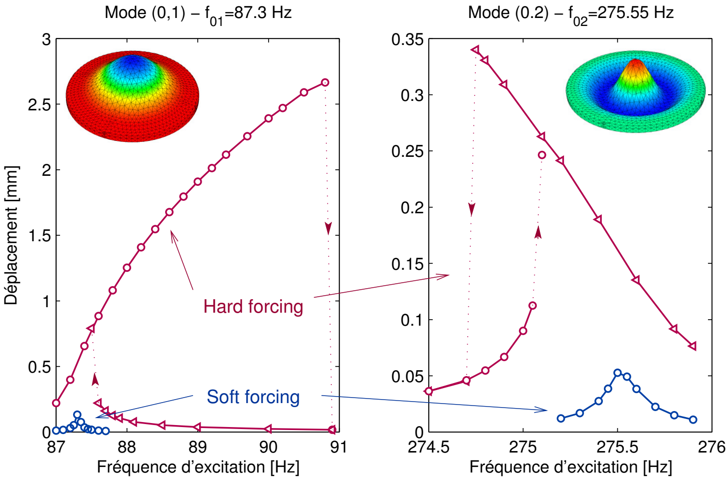
- Resonance curve: constant forcing amplitude, frequency sweep

Periodic regime analysis



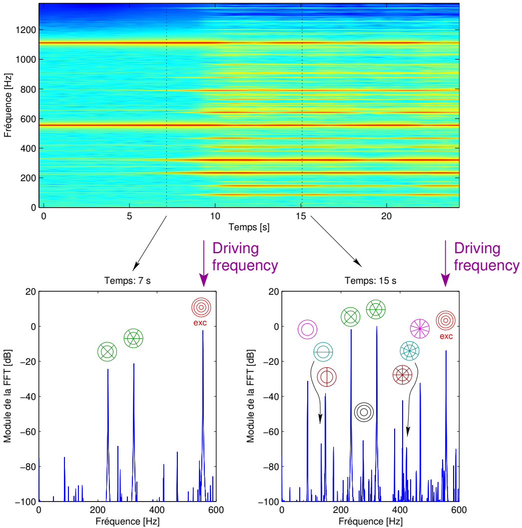
- Resonance curve: constant forcing amplitude, frequency sweep
- the resonance frequency / free oscillations frequency depend on the amplitude

Hardening or softening behaviour



▷ **The trend of nonlinearity depends on the considered mode**

Quasi-periodic regime analysis

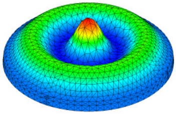


Internal resonances between modes

Forcing at center:

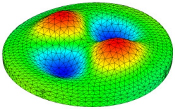
$f_{exc} \simeq f_{03}$

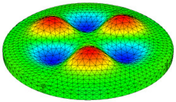
Mode (0,3)



Mode (2,1) $+$ Mode (3,1)

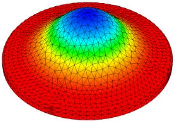
f_{21} f_{31}

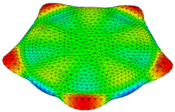




Mode (0,1) $+$ Mode (5,0)

f_{01} f_{50}





$$f_{03} \simeq f_{21} + f_{31} \simeq f_{01} + f_{50} \dots$$

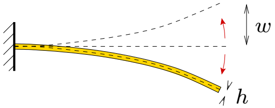
[Chaigne *et al.*, Acoust. Sci. & Tech., 2005]

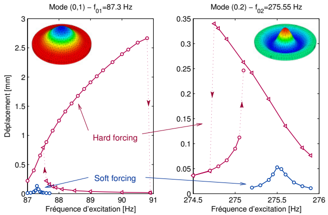
Couplings are governed by the **natural frequencies values**

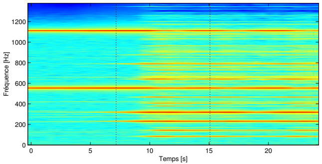
Motivations of this talk

▷ **Framework:**

- large amplitude non-linear vibrations of slender structures
- Non-linear vibrations:
  ⇝ jump phenomena
  ⇝ harmonic distortion
  ⇝ modal interactions



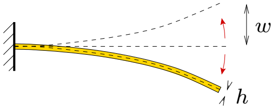


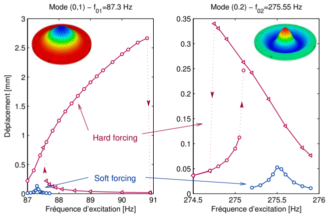


Motivations of this talk

▷ **Framework:**

- large amplitude non-linear vibrations of slender structures
- Non-linear vibrations:
  - ⇝ jump phenomena
  - ⇝ harmonic distortion
  - ⇝ modal interactions

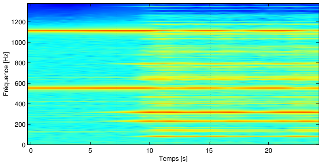




▷ **An overview of nonlinear modes:**

an efficient tool to

- analyse the non-linear vibratory regimes
- derive reduced order models



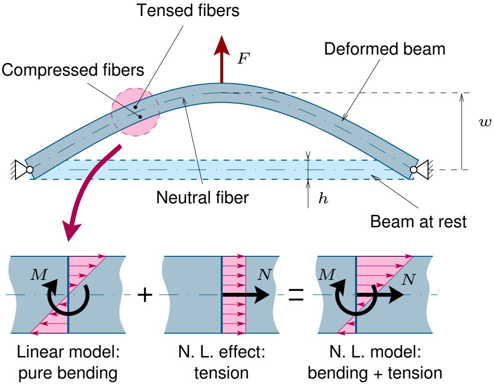
Outline

- ▷ **Introduction**
- ▷ **Models**
- ▷ **Nonlinear modes**
- ▷ **Numerical continuation**
- ▷ **Normal forms**
  - Internal resonances and resonant terms
  - Framework
  - Applications, validity
- ▷ **Conclusions**

Outline

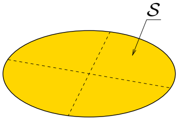
Mechanism of geometrical nonlinearities

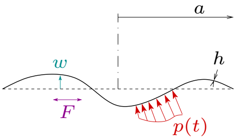
String effect in a beam



Increase of length $\Rightarrow$ increase of tension $\Rightarrow$ axial / bending coupling

Von-Kármán equations





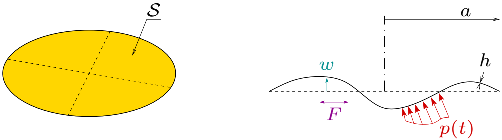
▷ **Nonlinerities source**

- In-plane / bending coupling

  ⇝ change of metrics of the neutral surface

  ⇝ membrane force field

Von-Kármán equations



▷ **Nonlinerities source**

- In-plane / bending coupling

  ⇝ change of metrics of the neutral surface

  ⇝ membrane force field

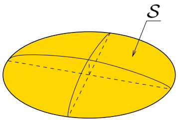
▷ **Plate model** [von Kármán 1910], [Herrmann 1955], [Thomas, Touzé *et al.* 2002-]

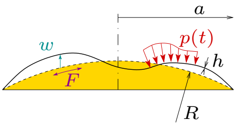
transverse: $$D\Delta\Delta w + \rho h \ddot{w} = L(w,F) - c\dot{w} + p,$$

membrane: $$\Delta\Delta F = -\frac{Eh}{2} L(w,w).$$

- nonlinear in-plane / bending coupling (geometrical N.L.)

Von-Kármán equations





▷ **Nonlinerities source**

- In-plane / bending coupling
  - ⇝ change of metrics of the neutral surface
  - ⇝ membrane force field

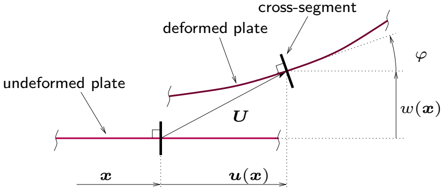
▷ **Spherical cap model** [Donnell 1934], [Mushtari & Galimov 1961], [Thomas, Touzé 2005]

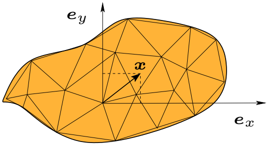
transverse: $$D\Delta\Delta w + \frac{1}{R}\Delta F + \rho h \ddot{w} = L(w,F) - c\dot{w} + p,$$

membrane: $$\Delta\Delta F - \frac{Eh}{R}\Delta w = -\frac{Eh}{2}L(w,w).$$

- nonlinear in-plane / bending coupling (geometrical N.L.)
- linear in-plane / bending coupling (curvature)

The problems to solve



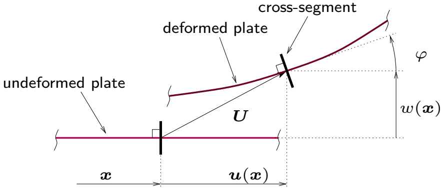


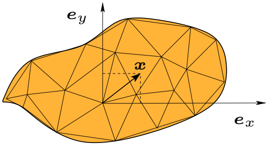
▷ **Analytical models**

- continuous unknowns $w(\boldsymbol{x})$, $u(\boldsymbol{x})$, $F(\boldsymbol{x})\dots$
- nonlinear partial differential equations: infinite dimension

$$\begin{cases} \rho h \ddot{w} + D\Delta\Delta w = L(w, F) + q \\ \Delta\Delta F = -\dfrac{Eh}{2} L(w, w) \end{cases}$$

The problems to solve





▷ Analytical models

- continuous unknowns $w(\boldsymbol{x})$, $u(\boldsymbol{x})$, $F(\boldsymbol{x})\dots$
- nonlinear partial differential equations: infinite dimension

$$\begin{cases} \rho h \ddot{w} + D\Delta\Delta w = L(w, F) + q \\ \Delta\Delta F = -\dfrac{Eh}{2} L(w, w) \end{cases}$$

▷ Finite elements models

- vector unknowns $\boldsymbol{U} = [u_1\, w_1\, u_2\, w_2 \dots w_N]^{\mathrm{t}}$
- nonlinear ordinary differential equations : large dimension $(N > 1000)$

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} + \boldsymbol{f}_{nl}(\boldsymbol{U}) = \boldsymbol{f}_e$$

Modal reduced order models

▷ K **eigenmodes**

⇝ We choose K eigenmodes $(\omega_k, \Phi_k(\boldsymbol{x}))$ or $(\omega_k, \boldsymbol{\Phi}_k)$, $\quad k = 1, \ldots, K$

(A truncated orthogonal basis of the solution space)

Modal reduced order models

▷ K **eigenmodes**

⇝ We choose K eigenmodes $(\omega_k, \Phi_k(\boldsymbol{x}))$ or $(\omega_k, \boldsymbol{\Phi}_k)$, $\quad k = 1, \dots, K$

(A truncated orthogonal basis of the solution space)

▷ **Modal expansion**

$$w(\boldsymbol{x}, t) = \sum_{i=1}^{K} \underbrace{\Phi_k(\boldsymbol{x})}_{\text{space}} \underbrace{q_k(t)}_{\text{time}} \qquad \text{or} \qquad \boldsymbol{U}(t) = \sum_{k=1}^{K} \underbrace{\boldsymbol{\Phi}_k}_{\text{space}} \underbrace{q_k(t)}_{\text{time}}$$

⇝ K unknown modal coordinates $q_k(t)$, $\quad k = 1, \dots, K$

Modal reduced order models

▷ K **eigenmodes**

⇝ We choose K eigenmodes $(\omega_k, \Phi_k(\boldsymbol{x}))$ or $(\omega_k, \boldsymbol{\Phi}_k)$, $\quad k = 1, \dots, K$

(A truncated orthogonal basis of the solution space)

▷ **Modal expansion**

$$w(\boldsymbol{x}, t) = \sum_{i=1}^{K} \underbrace{\Phi_k(\boldsymbol{x})}_{\text{space}} \underbrace{q_k(t)}_{\text{time}} \qquad \text{or} \qquad \boldsymbol{U}(t) = \sum_{k=1}^{K} \underbrace{\boldsymbol{\Phi}_k}_{\text{space}} \underbrace{q_k(t)}_{\text{time}}$$

⇝ K unknown modal coordinates $q_k(t)$, $\quad k = 1, \dots, K$

$$\Downarrow$$

$$\boxed{\ddot{q}_k + 2\mu_k \dot{q}_k + \omega_k^2 q_k = F_k}$$

- Linear uncoupled time ODEs

Modal reduced order models

▷ K **eigenmodes**

⇝ We choose K eigenmodes $(\omega_k, \Phi_k(\boldsymbol{x}))$ or $(\omega_k, \boldsymbol{\Phi}_k)$, $\quad k = 1, \dots, K$

(A truncated orthogonal basis of the solution space)

▷ **Modal expansion**

$$w(\boldsymbol{x}, t) = \sum_{i=1}^{K} \underbrace{\Phi_k(\boldsymbol{x})}_{\text{space}} \underbrace{q_k(t)}_{\text{time}} \qquad \text{or} \qquad \boldsymbol{U}(t) = \sum_{k=1}^{K} \underbrace{\boldsymbol{\Phi}_k}_{\text{space}} \underbrace{q_k(t)}_{\text{time}}$$

⇝ K unknown modal coordinates $q_k(t)$, $\quad k = 1, \dots, K$

$$\Downarrow$$

$$\boxed{\ddot{q}_k + 2\mu_k \dot{q}_k + \omega_k^2 q_k + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = F_k}$$

- Linear uncoupled time ODEs
- Cubic non-linear terms (geometrical N.L.)

Modal reduced order models

▷ K **eigenmodes**

⇝ We choose K eigenmodes $(\omega_k, \Phi_k(\boldsymbol{x}))$ or $(\omega_k, \boldsymbol{\Phi}_k)$, $\quad k = 1, \ldots, K$

(A truncated orthogonal basis of the solution space)

▷ **Modal expansion**

$$w(\boldsymbol{x}, t) = \sum_{i=1}^{K} \underbrace{\Phi_k(\boldsymbol{x})}_{\text{space}} \underbrace{q_k(t)}_{\text{time}} \qquad \text{or} \qquad \boldsymbol{U}(t) = \sum_{k=1}^{K} \underbrace{\boldsymbol{\Phi}_k}_{\text{space}} \underbrace{q_k(t)}_{\text{time}}$$

⇝ K unknown modal coordinates $q_k(t)$, $\quad k = 1, \ldots, K$

$$\Downarrow$$

$$\boxed{\ddot{q}_k + 2\mu_k \dot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = F_k}$$

- Linear uncoupled time ODEs
- Cubic non-linear terms (geometrical N.L.)
- Quadratic non-linear terms (curvature + geometrical N.L.)

Conclusions on models

▷ **Generic dynamical system (DS)**

(for any thin structure in moderate rotation)

$$\ddot{q}_k + 2\mu_k \dot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = F_k$$

Conclusions on models

▷ **Generic dynamical system (DS)**

(for any thin structure in moderate rotation)

$$\ddot{q}_k + 2\mu_k \dot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = F_k$$

▷ **Solving**

- Analytical methods, time integration, numerical continuation
- The main issue: the **truncation** of the modal basis

  ⇝ a solution: the **nonlinear mode** concept

Outline

Nonlinear modes: main goals

Extend the ("linear") eigenmode concept to the nonlinear range

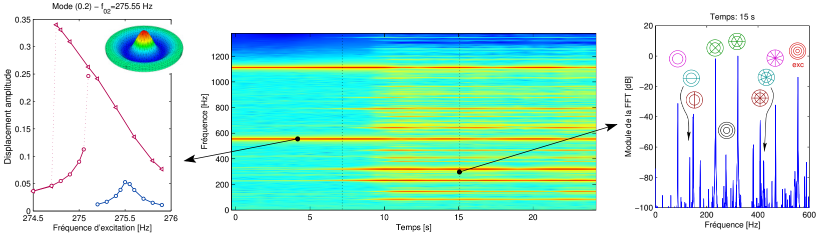
- **Produce reduced-order models**
- **Analyse the nonlinear vibratory regimes**

Nonlinear modes: main goals

Extend the ("linear") eigenmode concept to the nonlinear range

▷ **Produce reduced-order models**

▷ **Analyse the nonlinear vibratory regimes**



Periodic regime *Quasi–periodic regime*

- Hard-soft properties of modes, dependence of deformed shapes on the amplitude

  ⇝ 1 mode models ?

- Modal interactions (N modes), internal resonances

  ⇝ N mode models ?

Basics of “linear” modes

▷ **Solutions of the linear and undamped problems**

$$D\Delta\Delta\Phi - \rho h\omega^2\Phi = 0 \qquad \text{or} \qquad \left[\boldsymbol{K} - \omega^2\boldsymbol{M}\right]\boldsymbol{\Phi} = \boldsymbol{0}$$

⇝ eigenvalue problems of solutions (ω_i, Φ_i) or $(\omega_i, \boldsymbol{\Phi}_i)$

⇝ the eigenvectors $\{\Phi_i(\boldsymbol{x})\}$, $\{\boldsymbol{\Phi}_i\}$ are an orthogonal basis of the solution space

Basics of "linear" modes

▷ **Solutions of the linear and undamped problems**

$$D\Delta\Delta\Phi - \rho h\omega^2\Phi = 0 \qquad \text{or} \qquad \left[\boldsymbol{K} - \omega^2\boldsymbol{M}\right]\boldsymbol{\Phi} = \boldsymbol{0}$$

⇝ eigenvalue problems of solutions (ω_i, Φ_i) or $(\omega_i, \boldsymbol{\Phi}_i)$

⇝ the eigenvectors $\{\Phi_i(\boldsymbol{x})\}$, $\{\boldsymbol{\Phi}_i\}$ are an orthogonal basis of the solution space

▷ **Linear dynamics**

$$\begin{cases} w(\boldsymbol{x},t) = \displaystyle\sum_{i=1}^{N} \Phi_i(\boldsymbol{x}) q_i(t) \\ \boldsymbol{U}(t) = \displaystyle\sum_{i=1}^{N} \boldsymbol{\Phi}_i q_i(t) \end{cases} \Rightarrow \boxed{\underset{\{1,\ldots N\}}{\forall i} \quad \ddot{q}_i + \omega_i^2 q_i = 0}$$

⇝ a set of independent oscillators

⇝ with periodic (sine) time evolution

Basics of “linear” modes

▷ **Solutions of the linear and undamped problems**

$$D\Delta\Delta\Phi - \rho h \omega^2 \Phi = 0 \qquad \text{or} \qquad \left[\boldsymbol{K} - \omega^2 \boldsymbol{M}\right] \boldsymbol{\Phi} = \boldsymbol{0}$$

⇝ eigenvalue problems of solutions (ω_i, Φ_i) or $(\omega_i, \boldsymbol{\Phi}_i)$

⇝ the eigenvectors $\{\Phi_i(\boldsymbol{x})\}$, $\{\boldsymbol{\Phi}_i\}$ are an orthogonal basis of the solution space

▷ **Linear dynamics**

$$\begin{cases} w(\boldsymbol{x}, t) = \displaystyle\sum_{i=1}^{N} \Phi_i(\boldsymbol{x}) q_i(t) \\ \boldsymbol{U}(t) = \displaystyle\sum_{i=1}^{N} \boldsymbol{\Phi}_i q_i(t) \end{cases} \Rightarrow \boxed{\underset{\{1,\dots N\}}{\forall i} \quad \ddot{q}_i + \omega_i^2 q_i = 0}$$

⇝ a set of independent oscillators

⇝ with periodic (sine) time evolution

▷ **In the phase space**

$$\begin{cases} \dot{q}_i = v_i \\ \dot{v}_i = -\omega_i^2 q_i \end{cases}$$

⇝ the solution $(q_1, \dots q_N, v_1, \dots v_N)$ lives in a $2N$-dimensional space

Geometry of "linear" modes

▷ **Two features**

Free conservative vibrations, motion initiated on a particular mode:

- **periodic** oscillations
- **invariance** of motion (because of the **orthogonality**)

Geometry of “linear” modes

▷ **Two features**

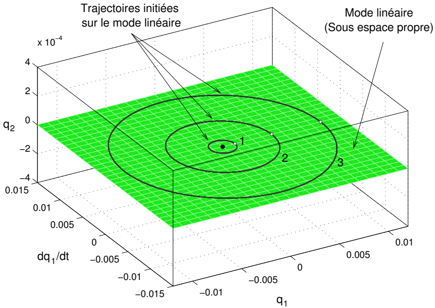
Free conservative vibrations, motion initiated on a particular mode:

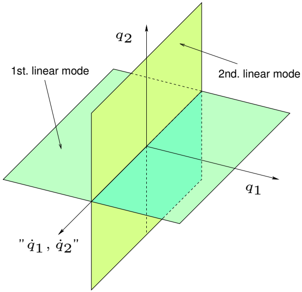
- **periodic** oscillations
- **invariance** of motion (because of the **orthogonality**)

▷ **In the phase space**

⇝ elliptic trajectories in planes (q_i, v_i)

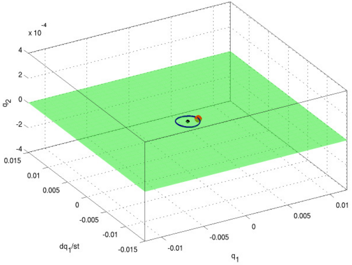
⇝ a “linear” mode is a plane eigen-subspace

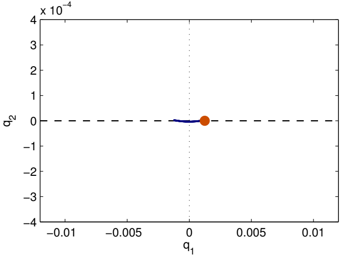




The nonlinear case (1)

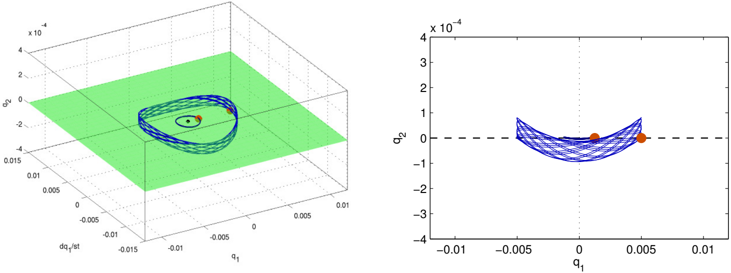
$$\ddot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = 0$$





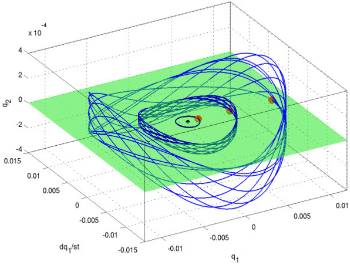
The nonlinear case (1)

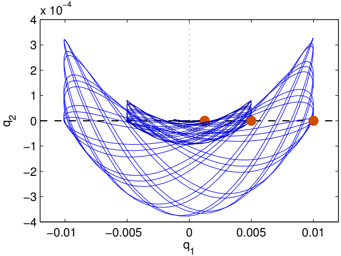
$$\ddot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = 0$$



The nonlinear case (1)

$$\ddot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = 0$$

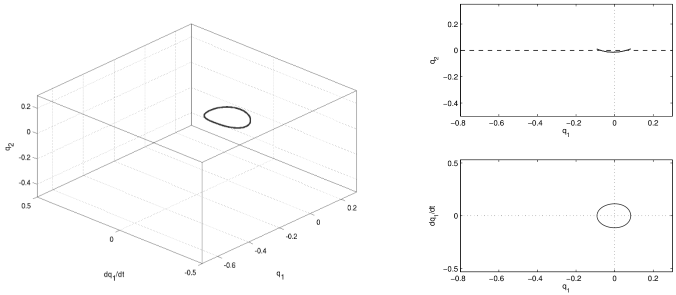




Coupling terms ⇒

- the trajectories are not periodical
- a trajectory initiated in a "linear" eigen-plane is not invariant

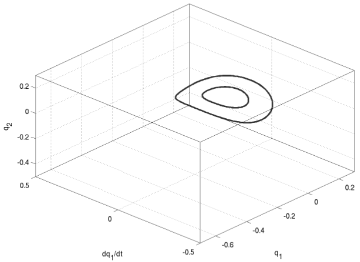
The nonlinear case (2)

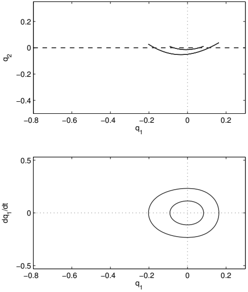


▷ **Periodic orbits**

"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

The nonlinear case (2)

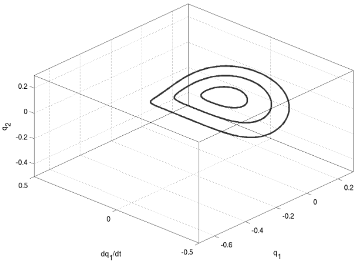


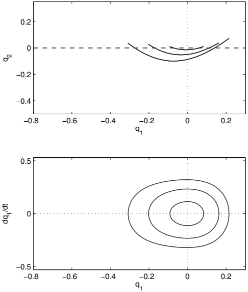


▷ **Periodic orbits**

"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

The nonlinear case (2)

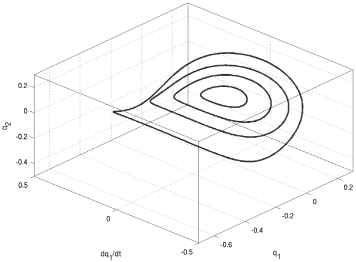


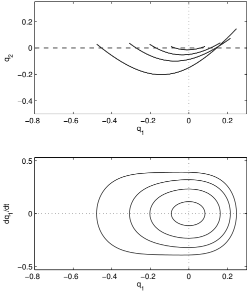


▷ **Periodic orbits**

"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

The nonlinear case (2)

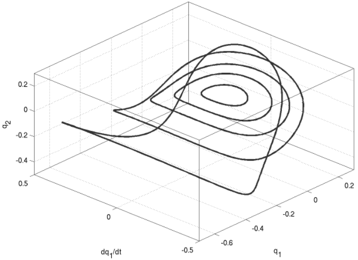


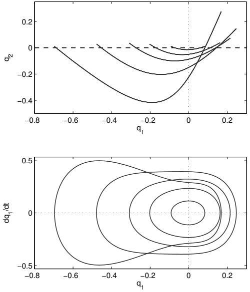


▷ **Periodic orbits**

"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

The nonlinear case (2)

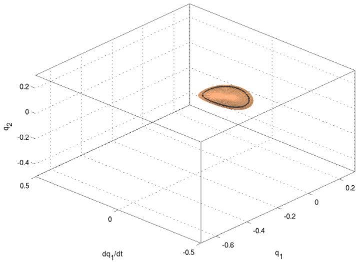


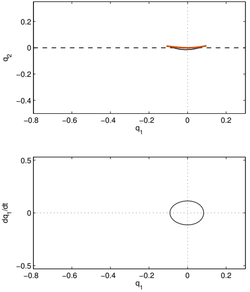


▷ **Periodic orbits**

"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

The nonlinear case (2)





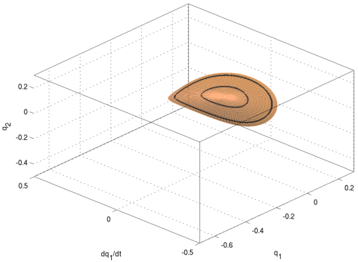
▷ **Periodic orbits**

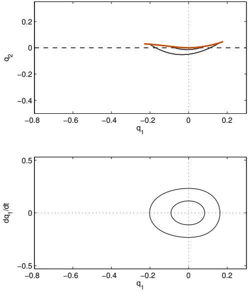
"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

▷ **Invariant manifold**

Those periodic orbits are invariant and thus define an invariant manifold

The nonlinear case (2)





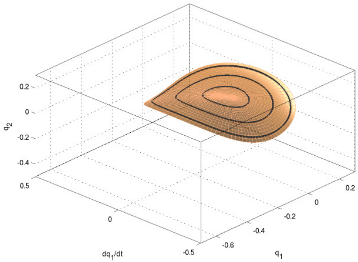
▷ **Periodic orbits**

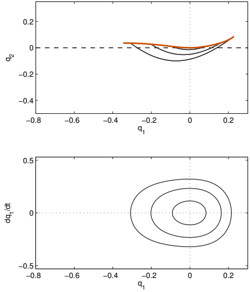
"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

▷ **Invariant manifold**

Those periodic orbits are invariant and thus define an invariant manifold

The nonlinear case (2)





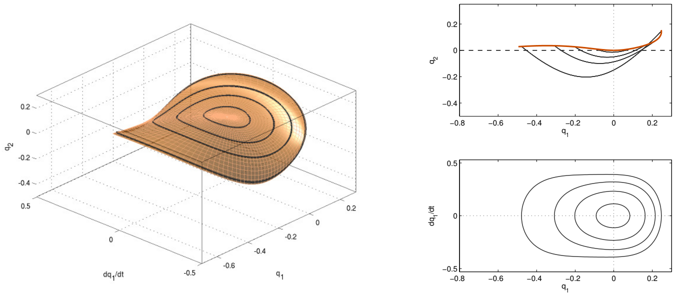
- **Periodic orbits**

  "Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

- **Invariant manifold**

  Those periodic orbits are invariant and thus define an invariant manifold

The nonlinear case (2)



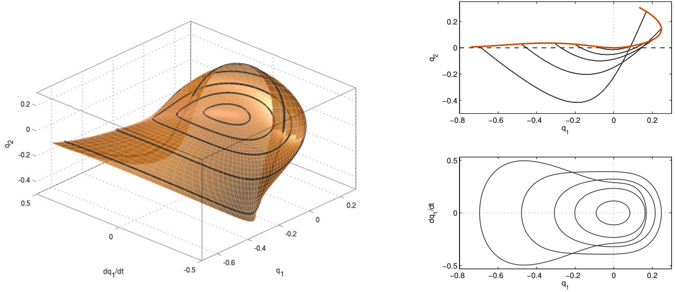
▷ **Periodic orbits**

"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

▷ **Invariant manifold**

Those periodic orbits are invariant and thus define an invariant manifold

The nonlinear case (2)



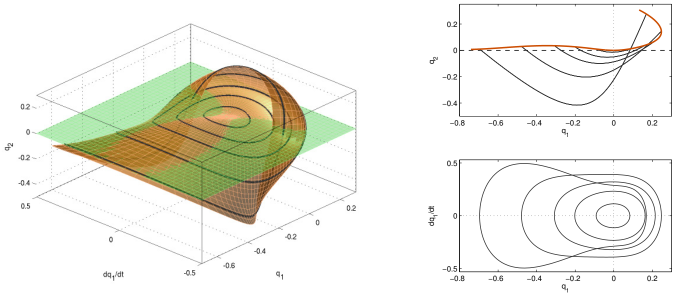
▷ **Periodic orbits**

"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

▷ **Invariant manifold**

Those periodic orbits are invariant and thus define an invariant manifold

The nonlinear case (2)



▷ **Periodic orbits**

"Any $2N$ dimensional conservative dynamical system has at least N families of periodic orbits in the vicinity of a stable equilibrium point" [Lyapunov 1907]

▷ **Invariant manifold**

Those periodic orbits are invariant and thus define an invariant manifold

Nonlinear modes (NLM): definitions

▷ **Two features preserved in the nonlinear case**

- "A NLM is a family of **periodic** orbits"
  [Lyapunov 1907, Rosenberg, 1960, Vakakis, Manevitch, Mikhlin, Kerschen 1996-]
- "A NLM is a 2D **invariant** manifold, tangent at the origin to the linear eigen-planes"
  [Shaw, Pierre, 1991-]

Nonlinear modes (NLM): definitions

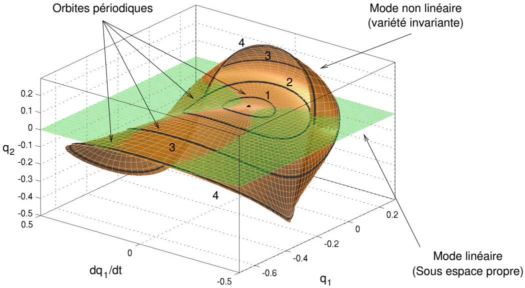
▷ Two features preserved in the nonlinear case

- "A NLM is a family of **periodic** orbits"

  [Lyapunov 1907, Rosenberg, 1960, Vakakis, Manevitch, Mikhlin, Kerschen 1996-]
- "A NLM is a 2D **invariant** manifold, tangent at the origin to the linear eigen-planes"

  [Shaw, Pierre, 1991-]

▷ In the phase space



Equivalence of the definitions

"invariant manifold" more general than "periodic orbits"

▷ **For conservative systems**

$$\ddot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = 0$$

Periodic orbits ⇔ invariant manifold

▷ **For dissipative systems**

$$\ddot{q}_k + 2\mu_k \dot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = 0$$

- **no periodic orbits** are solutions (in free vibrations)
- with modal (diagonal) damping, there exist **invariant manifolds** distinct from the associated conservative ones [Touzé, Amabili 2006]

  (for linear systems, the eigen-planes are the same)
- with non-diagonal damping ? Probably the same with an eigenspectrum $\in \mathbb{C}$

  (for linear systems, the modes are complex and the eigenplanes are distincts)

How to compute them

Three main families of strategies

▷ **Computation of periodic orbits**

- Analytical methods [Vakakis *et al.*, Springer 2008]
- Numerical continuation

  ⇝ see in the following

▷ **Computation of the invariant manifold**

- Analytical and numerical methods

  [Shaw, Pierre 1991–], [Blanc, Touzé *et al.* MSSP 2013]

▷ **Using normal forms**

[Jézéquel & Lamarque JSV 1991], [Touzé, Thomas, Amabili JSV 2004, 2006]

⇝ see in the following

Outline

- Introduction
- Models
- Nonlinear modes
- **Numerical continuation**
- Normal forms
  - Internal resonances and resonant terms
  - Framework
  - Applications, validity
- Conclusions

Continuation of fixed points

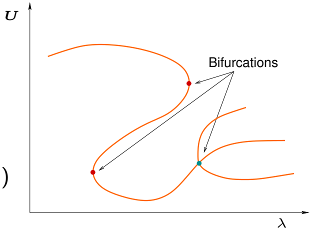
▷ **Problem formulation** (ex: buckling)

$$\boldsymbol{R}(\boldsymbol{U}, \lambda) = \boldsymbol{0},$$

$$\boldsymbol{R} \in \mathbb{R}^N, \quad \boldsymbol{U} = (U_1, \dots, U_N)^{\mathrm{t}} \in \mathbb{R}^N,$$

with $\boldsymbol{U}$: unknown; λ: control parameter,

$\Rightarrow$ curves $\boldsymbol{U} = \boldsymbol{f}(\lambda)$ (implicit function theorem)



Continuation of fixed points

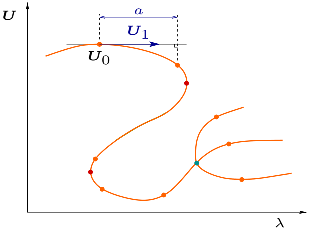
▷ **Problem formulation** (ex: buckling)

$$\boldsymbol{R}(\boldsymbol{U}, \lambda) = \boldsymbol{0},$$

$$\boldsymbol{R} \in \mathbb{R}^N, \quad \boldsymbol{U} = (U_1, \ldots, U_N)^{\mathrm{t}} \in \mathbb{R}^N,$$

with $\boldsymbol{U}$: unknown; λ: control parameter,

$\Rightarrow$ curves $\boldsymbol{U} = \boldsymbol{f}(\lambda)$ (implicit function theorem)



▷ **Path parametrization**

$\rightsquigarrow a$: arclength, pseudo arclength ...

Continuation of fixed points

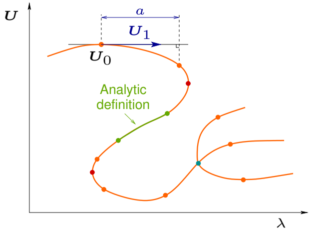
▷ **Problem formulation** (ex: buckling)

$$\boldsymbol{R}(\boldsymbol{U}, \lambda) = \boldsymbol{0},$$

$$\boldsymbol{R} \in \mathbb{R}^N, \quad \boldsymbol{U} = (U_1, \ldots, U_N)^{\mathrm{t}} \in \mathbb{R}^N,$$

with $\boldsymbol{U}$: unknown; λ: control parameter,

$\Rightarrow$ curves $\boldsymbol{U} = \boldsymbol{f}(\lambda)$ (implicit function theorem)



▷ **Path parametrization**

$\rightsquigarrow$ a: arclength, pseudo arclength ...

▷ **Newton-Raphson method**

$\rightsquigarrow$ classical [Crisfield Wiley 1996], AUTO software...

Continuation of fixed points

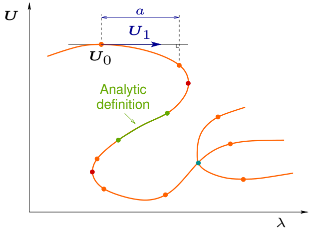
▷ **Problem formulation** (ex: buckling)

$$\boldsymbol{R}(\boldsymbol{U},\lambda)=\boldsymbol{0},$$

$$\boldsymbol{R}\in\mathbb{R}^N,\quad \boldsymbol{U}=(U_1,\ldots,U_N)^{\mathrm{t}}\in\mathbb{R}^N,$$

with $\boldsymbol{U}$: unknown; λ: control parameter,

$\Rightarrow$ curves $\boldsymbol{U}=\boldsymbol{f}(\lambda)$ (implicit function theorem)



▷ **Path parametrization**

⇝ a: arclength, pseudo arclength ...

▷ **Newton-Raphson method**

⇝ classical [Crisfield Wiley 1996], AUTO software...

▷ **The Asymptotic Numerical Method (ANM)** [Potier-Ferry, Cochelin *et al.*, 1990–]

- Power series expansions: $\boldsymbol{U}(a)=\boldsymbol{U}_0+\boldsymbol{U}_1 a+\ldots+\ldots+\boldsymbol{U}_n a^n$
- Automatic stepping, very few control parameters,
- Graphical tool coded in Matlab [Arquier, 2007], [Cochelin & Vergez, 2009]
- Home made ANM code

Nonlinear dynamics: continuation of periodic solutions

$$\left\{ \begin{array}{l} \text{dynamical system} \\ T\text{-periodic solution} \end{array} \right. \quad \Rightarrow \quad \text{algebraic problem}$$

▷ **Several methods**

- Shooting method [Kerschen *et al.* MSSP 2009]
- Collocation method [AUTO software]
- Finite differences [Cochelin CS 2006]
- and. . .

Nonlinear dynamics: continuation of periodic solutions

$$\left\{\begin{array}{l}\text{dynamical system} \\ T\text{-periodic solution}\end{array}\right. \quad \Rightarrow \quad \text{algebraic problem}$$

▷ **Several methods**

- Shooting method [Kerschen *et al.* MSSP 2009]
- Collocation method [AUTO software]
- Finite differences [Cochelin CS 2006]
- and...

▷ **The harmonic balance method (HBM)**

- Fourier series decomposition of the unknowns

$$\boldsymbol{q}(t) = \boldsymbol{q}^{(0)} + \sum_{h=1}^{H} \boldsymbol{q}^{(hc)} \cos h\Omega t + \sum_{h=1}^{H} \boldsymbol{q}^{(hs)} \sin h\Omega t$$

- algebraic set of equations in $\boldsymbol{q}^{(0)}$, $\boldsymbol{q}^{(hc)}$, $\boldsymbol{q}^{(hs)}$

Nonlinear dynamics: continuation of periodic solutions

$$\left\{ \begin{array}{l} \text{dynamical system} \\ T\text{-periodic solution} \end{array} \right. \quad \Rightarrow \quad \text{algebraic problem}$$

▷ **Several methods**

- Shooting method [Kerschen *et al.* MSSP 2009]
- Collocation method [AUTO software]
- Finite differences [Cochelin CS 2006]
- and...

▷ **The harmonic balance method (HBM)**

- Fourier series decomposition of the unknowns

$$\boldsymbol{q}(t) = \boldsymbol{q}^{(0)} + \sum_{h=1}^{H} \boldsymbol{q}^{(hc)} \cos h\Omega t + \sum_{h=1}^{H} \boldsymbol{q}^{(hs)} \sin h\Omega t$$

- algebraic set of equations in $\boldsymbol{q}^{(0)}$, $\boldsymbol{q}^{(hc)}$, $\boldsymbol{q}^{(hs)}$

▷ **The Asymptotic Numerical Method (ANM)** [Potier-Ferry, Cochelin *et al.*, 1990–]

Nonlinear dynamics: continuation of periodic solutions

$$\left\{ \begin{array}{l} \text{dynamical system} \\ T\text{-periodic solution} \end{array} \right. \quad \Rightarrow \quad \text{algebraic problem}$$

▷ **Several methods**

- Shooting method [Kerschen *et al.* MSSP 2009]
- Collocation method [AUTO software]
- Finite differences [Cochelin CS 2006]
- and...

▷ **The harmonic balance method (HBM)**

- Fourier series decomposition of the unknowns

$$\boldsymbol{q}(t) = \boldsymbol{q}^{(0)} + \sum_{h=1}^{H} \boldsymbol{q}^{(hc)} \cos h\Omega t + \sum_{h=1}^{H} \boldsymbol{q}^{(hs)} \sin h\Omega t$$

- algebraic set of equations in $\boldsymbol{q}^{(0)}$, $\boldsymbol{q}^{(hc)}$, $\boldsymbol{q}^{(hs)}$

▷ **The Asymptotic Numerical Method (ANM)** [Potier-Ferry, Cochelin *et al.*, 1990–]

▷ **Frequency domain stability computation: Hill method** [Lazarus & Thomas, 2010]

Nonlinear dynamics: continuation of periodic solutions

$$\left\{\begin{array}{l}\text{dynamical system}\\ T\text{-periodic solution}\end{array}\right. \quad \Rightarrow \quad \text{algebraic problem}$$

▷ **Several methods**

- Shooting method [Kerschen *et al.* MSSP 2009]
- Collocation method [AUTO software]
- Finite differences [Cochelin CS 2006]
- and...

▷ **The harmonic balance method (HBM)**

- Fourier series decomposition of the unknowns

$$\boldsymbol{q}(t) = \boldsymbol{q}^{(0)} + \sum_{h=1}^{H} \boldsymbol{q}^{(hc)} \cos h\Omega t + \sum_{h=1}^{H} \boldsymbol{q}^{(hs)} \sin h\Omega t$$

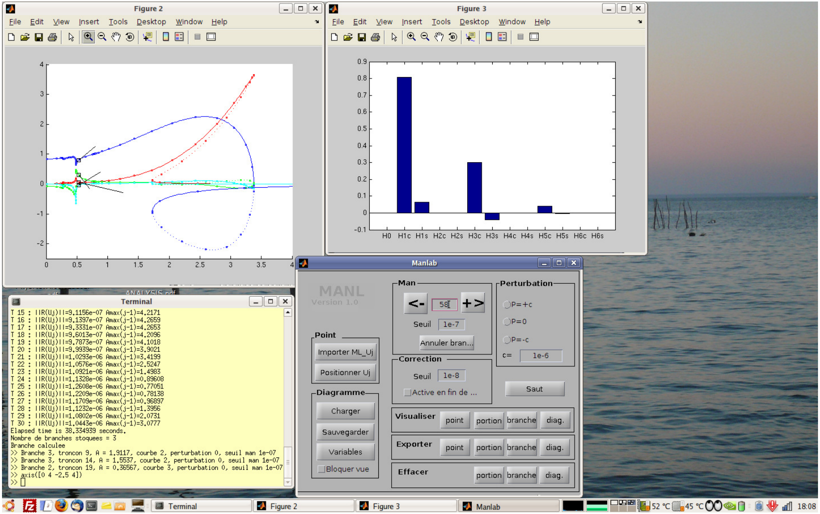
- algebraic set of equations in $\boldsymbol{q}^{(0)}$, $\boldsymbol{q}^{(hc)}$, $\boldsymbol{q}^{(hs)}$

▷ **The Asymptotic Numerical Method (ANM)** [Potier-Ferry, Cochelin *et al.*, 1990–]

▷ **Frequency domain stability computation: Hill method** [Lazarus & Thomas, 2010]

HBM/ANM/Hill: a complete tool for continuation of periodic solutions

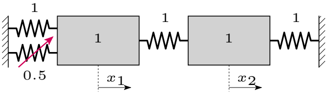
Manlab screenshot



http://manlab.lma.cnrs-mrs.fr/

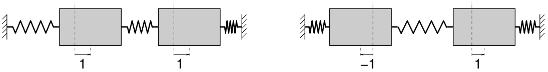
An example

$$\begin{cases} \ddot{u}_1 + 2u_1 - u_2 + 0.5u_1^3 = 0 \\ \ddot{u}_2 + 2u_2 - u_1 = 0 \end{cases}$$



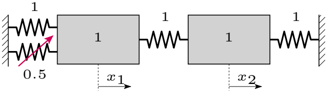
[Kerschen *et al.* MSSP 2009]

▷ **Modes “linéaires”**



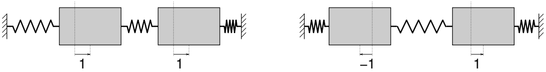
An example

$$\begin{cases} \ddot{u}_1 + 2u_1 - u_2 + 0.5u_1^3 = 0 \\ \ddot{u}_2 + 2u_2 - u_1 = 0 \end{cases}$$

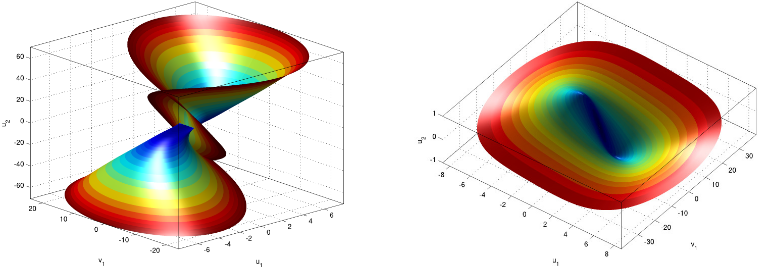


[Kerschen *et al.* MSSP 2009]

▷ **Modes "linéaires"**

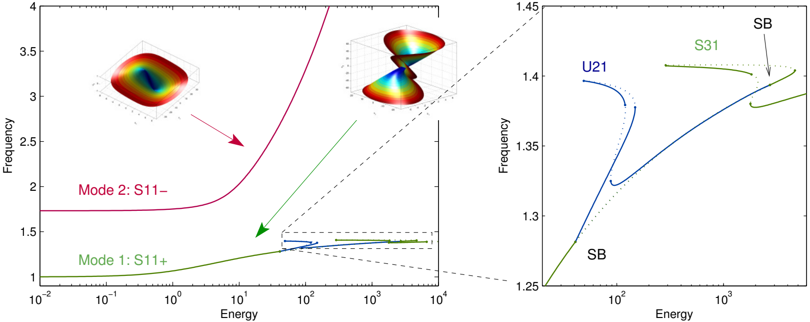


▷ **Modes non linéaires**



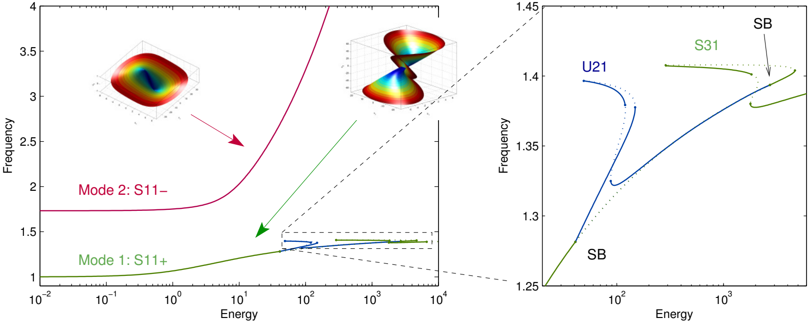
Continuation as a function of the energy + assembling of the orbits

Backbone curves ⇝ frequency-energy plot (FEP)



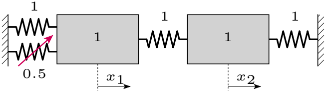
- Change of free oscillations frequency as a function of energy (amplitude)
  ⇝ mode 1 & 2 are hardening

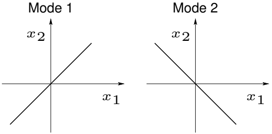
Backbone curves ⇝ frequency-energy plot (FEP)

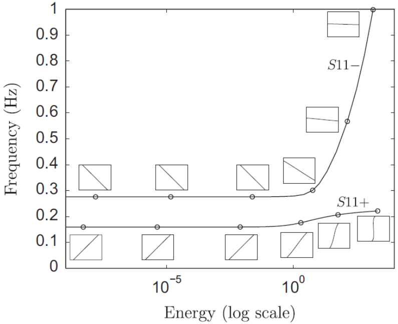


- Change of free oscillations frequency as a function of energy (amplitude)
  ⇝ mode 1 & 2 are hardening
- Presence of internal resonances (U21: $\omega_{nl2} \simeq 2\omega_{nl1}$; U31: $\omega_{nl2} \simeq 3\omega_{nl1}$):
  ⇝ instabilities, symmetry-breaking bifurcations,
  ⇝ non synchronous motion,
  ⇝ loops in the FEP, folds of the manifolds

The mode shapes also depend on the energy

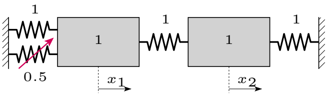


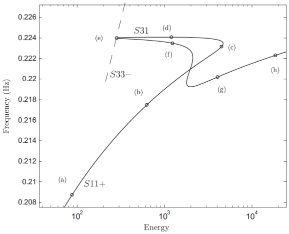


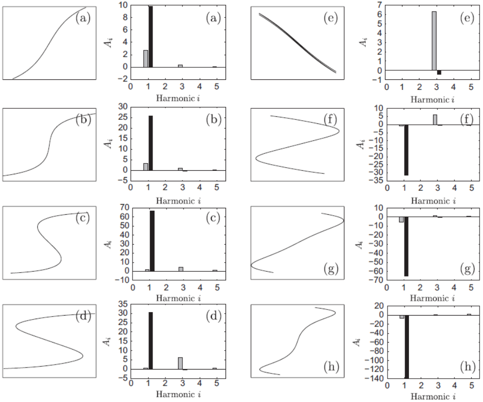


- localization of the energy

The mode shapes also depend on the energy





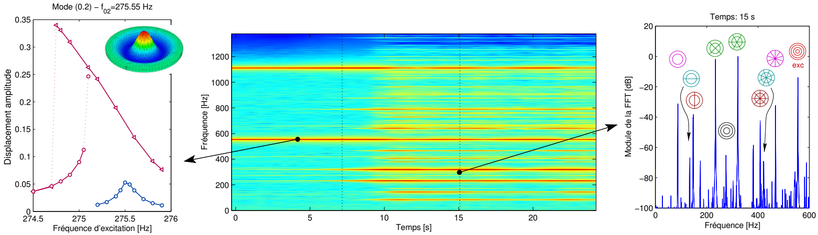


- localization of the energy
- non-synchronous motion for internal resonances

  ⇝ 3:1 int. resonance: mass 1 oscillates 3× faster than mass 2

Outline

Nonlinear model reduction



Periodic regime *Quasi–periodic regime*

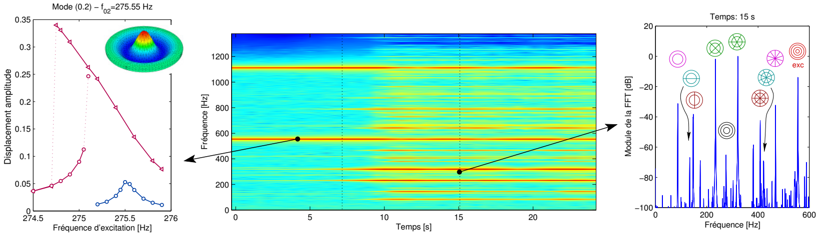
$$\ddot{q}_k + 2\mu_k \dot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = F_k$$

▷ **Some phenomena to describe**

- the resonance frequency depends on the amplitude
- energy transfers between "modes"

$\rightsquigarrow$ the nonlinear coupling terms are responsible

Nonlinear model reduction



Periodic regime *Quasi–periodic regime*

$$\ddot{q}_k + 2\mu_k \dot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \Gamma_{ijl}^k q_i q_j q_l = F_k$$

▷ **Some phenomena to describe**

- the resonance frequency depends on the amplitude
- energy transfers between "modes"

⇝ the nonlinear coupling terms are responsible

▷ **One central question**

⇝ which modes do we have to keep in the modal truncation ?

Resonant terms and internal resonances (1)

▷ **A two mode quadratic model**

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = \beta_1 q_1^2 + \beta_2 q_2^2 + \beta_3 q_1 q_2 \\ \ddot{q}_2 + \omega_2^2 q_2 = \beta_4 q_1^2 + \beta_5 q_2^2 + \beta_6 q_1 q_2 \end{cases}$$

Resonant terms and internal resonances (1)

▷ **A two mode quadratic model**

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = \beta_1 q_1^2 + \beta_2 q_2^2 + \beta_3 q_1 q_2 \\ \ddot{q}_2 + \omega_2^2 q_2 = \beta_4 q_1^2 + \beta_5 q_2^2 + \beta_6 q_1 q_2 \end{cases}$$

▷ **At first order**

$$q_1 = \cos \omega_1 t, \quad q_2 = \cos \omega_2 t$$

$$q_1^2 = \frac{1}{2}\left(1 + \cos 2\omega_1 t\right)$$ Harmonics $0 \quad 2\omega_1$

$$q_2^2 = \frac{1}{2}\left(1 + \cos 2\omega_2 t\right)$$ Harmonics $0 \quad 2\omega_2$

$$q_1 q_2 = \frac{1}{2}\left(\cos[\omega_1 + \omega_2]t + \cos[\omega_1 - \omega_2]t\right)$$ Harmonics $\omega_1 + \omega_2 \quad |\omega_1 - \omega_2|$

Resonant terms and internal resonances (1)

▷ **A two mode quadratic model**

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = \beta_1 q_1^2 + \beta_2 q_2^2 + \beta_3 q_1 q_2 \\ \ddot{q}_2 + \omega_2^2 q_2 = \beta_4 q_1^2 + \beta_5 q_2^2 + \beta_6 q_1 q_2 \end{cases}$$

▷ **At first order**

$$q_1 = \cos\omega_1 t, \quad q_2 = \cos\omega_2 t$$

$$q_1^2 = \frac{1}{2}(1+\cos 2\omega_1 t) \qquad \text{Harmonics} \quad 0 \quad 2\omega_1 = \omega_2$$

$$q_2^2 = \frac{1}{2}(1+\cos 2\omega_2 t) \qquad \text{Harmonics} \quad 0 \quad 2\omega_2$$

$$q_1 q_2 = \frac{1}{2}(\cos[\omega_1+\omega_2]t + \cos[\omega_1-\omega_2]t) \qquad \text{Harmonics} \quad \omega_1+\omega_2 \quad |\omega_1-\omega_2| = \omega_1$$

▷ **Quadratic internal resonance**

$$\omega_2 \simeq 2\omega_1$$

Resonant terms and internal resonances (1)

▷ **A two mode quadratic model**

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = \beta_1 q_1^2 + \cancel{\beta_2 q_2^2} + \cancel{\beta_3 q_1 q_2} \\ \ddot{q}_2 + \omega_2^2 q_2 = \cancel{\beta_4 q_1^2} + \cancel{\beta_5 q_2^2} + \beta_6 q_1 q_2 \end{cases}$$

▷ **At first order**

$$q_1 = \cos\omega_1 t, \quad q_2 = \cos\omega_2 t$$

$$q_1^2 = \frac{1}{2}(1+\cos 2\omega_1 t) \qquad \text{Harmonics} \quad 0 \quad 2\omega_1 = \omega_2$$

$$q_2^2 = \frac{1}{2}(1+\cos 2\omega_2 t) \qquad \text{Harmonics} \quad 0 \quad 2\omega_2$$

$$q_1 q_2 = \frac{1}{2}(\cos[\omega_1+\omega_2]t + \cos[\omega_1-\omega_2]t) \qquad \text{Harmonics} \quad \omega_1+\omega_2 \quad |\omega_1-\omega_2| = \omega_1$$

▷ **Quadratic internal resonance**

$$\omega_2 \simeq 2\omega_1$$

▷ **Resonant terms**

- they can be viewed as terms that **drive oscillators at their resonance**

Resonant terms and internal resonances (1)

▷ A two mode quadratic model

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = \beta_1 q_1^2 + \cancel{\beta_2 q_2^2} + \cancel{\beta_3 q_1 q_2} \\ \ddot{q}_2 + \omega_2^2 q_2 = \cancel{\beta_4 q_1^2} + \cancel{\beta_5 q_2^2} + \beta_6 q_1 q_2 \end{cases}$$

▷ At first order

$$q_1 = \cos\omega_1 t, \quad q_2 = \cos\omega_2 t$$

$$q_1^2 = \frac{1}{2}\left(1 + \cos 2\omega_1 t\right) \qquad \text{Harmonics} \quad 0 \quad 2\omega_1 = \omega_2$$

$$q_2^2 = \frac{1}{2}\left(1 + \cos 2\omega_2 t\right) \qquad \text{Harmonics} \quad 0 \quad 2\omega_2$$

$$q_1 q_2 = \frac{1}{2}\left(\cos[\omega_1 + \omega_2]t + \cos[\omega_1 - \omega_2]t\right) \qquad \text{Harmonics} \quad \omega_1 + \omega_2 \quad |\omega_1 - \omega_2| = \omega_1$$

▷ Quadratic internal resonance

$$\omega_2 \simeq 2\omega_1$$

▷ Resonant terms

- they can be viewed as terms that **drive oscillators at their resonance**
- they are linked to a particular internal resonance

Resonant terms and internal resonances (1)

▷ **A two mode quadratic model**

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = \beta_1 q_1^2 + \cancel{\beta_2 q_2^2} + \cancel{\beta_3 q_1 q_2} \\ \ddot{q}_2 + \omega_2^2 q_2 = \cancel{\beta_4 q_1^2} + \cancel{\beta_5 q_2^2} + \beta_6 q_1 q_2 \end{cases}$$

▷ **At first order**

$$q_1 = \cos \omega_1 t, \quad q_2 = \cos \omega_2 t$$

$$q_1^2 = \frac{1}{2}(1 + \cos 2\omega_1 t) \qquad \text{Harmonics} \quad 0 \quad 2\omega_1 = \omega_2$$

$$q_2^2 = \frac{1}{2}(1 + \cos 2\omega_2 t) \qquad \text{Harmonics} \quad 0 \quad 2\omega_2$$

$$q_1 q_2 = \frac{1}{2}(\cos[\omega_1 + \omega_2]t + \cos[\omega_1 - \omega_2]t) \qquad \text{Harmonics} \quad \omega_1 + \omega_2 \quad |\omega_1 - \omega_2| = \omega_1$$

▷ **Quadratic internal resonance**

$$\omega_2 \simeq 2\omega_1$$

▷ **Resonant terms**

- they can be viewed as terms that **drive oscillators at their resonance**
- they are linked to a particular internal resonance
- they are the skeleton of the dynamics and are responsible of the **energy transfers between modes**

Resonant terms and internal resonances (2)

▷ **Frequency relations for internal resonance**

- of order 2, linked to quadratic nonlinear terms:

$$\omega_2 = 2\omega_1, \quad \omega_3 = \omega_1 + \omega_2$$

- of order 3, linked to cubic nonlinear terms:

$$\omega_2 = \omega_1, \quad \omega_2 = 3\omega_1, \quad \omega_3 = \omega_1 + 2\omega_2 \quad \omega_4 = \omega_1 + \omega_2 + \omega_3$$

- of order N:

$$\omega_k = \sum_{i=1}^{N} m_i \omega_i, \quad m_i \in \mathbb{Z} \quad \sum_i |m_i| = N \geq 2$$

Resonant terms and internal resonances (2)

▷ **Frequency relations for internal resonance**

- of order 2, linked to quadratic nonlinear terms:

$$\omega_2 = 2\omega_1, \quad \omega_3 = \omega_1 + \omega_2$$

- of order 3, linked to cubic nonlinear terms:

$$\omega_2 = \omega_1, \quad \omega_2 = 3\omega_1, \quad \omega_3 = \omega_1 + 2\omega_2 \quad \omega_4 = \omega_1 + \omega_2 + \omega_3$$

- of order N:

$$\omega_k = \sum_{i=1}^{N} m_i \omega_i, \quad m_i \in \mathbb{Z} \quad \sum_i |m_i| = N \geq 2$$

▷ **Remark: some cubic terms are always resonant**

General model truncated to 1 oscillator: Duffing equation:

$$\ddot{q} + 2\mu\dot{q} + \omega_0^2 q + \beta q^2 + \Gamma q^3 = 0$$

$\rightsquigarrow q^3 = \frac{1}{4}\left(3\cos\omega_0 t + \cos 3\omega_0 t\right)$

$\rightsquigarrow q^3$ is **always resonant** (it is not linked to an internal resonance)

Normal form

A way to compute nonlinear modes

▷ **Modal model**

$$\underset{\{1,\dots,K\}}{\forall p} \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{K}\sum_{j\geq i}^{K} g_{ij}^p X_i X_j + \sum_{i=1}^{K}\sum_{j\geq i}^{K}\sum_{k\geq j}^{K} h_{ijk}^p X_i X_j X_k = 0.$$

Normal form

A way to compute nonlinear modes

▷ **Modal model**

$$\forall p_{\{1,\dots,K\}} \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{K}\sum_{j\geq i}^{K} g_{ij}^p X_i X_j + \sum_{i=1}^{K}\sum_{j\geq i}^{K}\sum_{k\geq j}^{K} h_{ijk}^p X_i X_j X_k = 0.$$

▷ **Principle** [Poincaré 1892, Dulac 1912, Jézéquel & Lamarque, 1991]

- **Nonlinear** polynomial change of variables ($Y_p = \dot{X}_p$, $S_p = \dot{R}_p$)

$$X_p = R_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}(a_{ij}^p R_i R_j + b_{ij}^p S_i S_j) + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N} r_{ijk}^p R_i R_j R_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N} u_{ijk}^p R_i S_j S$$

$$Y_p = S_p + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}^p R_i S_j + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N}\mu_{ijk}^p S_i S_j S_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N}\nu_{ijk}^p S_i R_j R_k$$

Normal form

A way to compute nonlinear modes

▷ **Modal model**

$$\underset{\{1,\dots,K\}}{\forall p} \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{K}\sum_{j\geq i}^{K} g_{ij}^p X_i X_j + \sum_{i=1}^{K}\sum_{j\geq i}^{K}\sum_{k\geq j}^{K} h_{ijk}^p X_i X_j X_k = 0.$$

▷ **Principle** [Poincaré 1892, Dulac 1912, Jézéquel & Lamarque, 1991]

- **Nonlinear** polynomial change of variables ($Y_p = \dot{X}_p$, $S_p = \dot{R}_p$)

$$X_p = R_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}(a_{ij}^p R_i R_j + b_{ij}^p S_i S_j) + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N} r_{ijk}^p R_i R_j R_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N} u_{ijk}^p R_i S_j S$$

$$Y_p = S_p + \sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_{ij}^p R_i S_j + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N} \mu_{ijk}^p S_i S_j S_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N} \nu_{ijk}^p S_i R_j R_k$$

- New **reduced** dynamical system in $(R_i, \dot{R}_i)$ ⇝ **normal form**

  ⇝ **without any non resonant term** that break the invariance

  ⇝ exact truncation

Normal form

A way to compute nonlinear modes

▷ **Modal model**

$$\forall p_{\{1,\dots,K\}} \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{K}\sum_{j\geq i}^{K} g_{ij}^p X_i X_j + \sum_{i=1}^{K}\sum_{j\geq i}^{K}\sum_{k\geq j}^{K} h_{ijk}^p X_i X_j X_k = 0.$$

▷ **Principle** [Poincaré 1892, Dulac 1912, Jézéquel & Lamarque, 1991]

- **Nonlinear** polynomial change of variables ($Y_p = \dot{X}_p$, $S_p = \dot{R}_p$)

$$X_p = R_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}(a_{ij}^p R_i R_j + b_{ij}^p S_i S_j) + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N} r_{ijk}^p R_i R_j R_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N} u_{ijk}^p R_i S_j S$$

$$Y_p = S_p + \sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_{ij}^p R_i S_j + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N} \mu_{ijk}^p S_i S_j S_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N} \nu_{ijk}^p S_i R_j R_k$$

- New **reduced** dynamical system in $(R_i, \dot{R}_i)$ ⇝ **normal form**

  ⇝ **without any non resonant term** that break the invariance

  ⇝ exact truncation

▷ **A general framework**

- Formal computation up to order 3 of the change of variables and of the normal dynamics [Touzé, Thomas, Chaigne 2004]
- **Automatic** and **a priori** writing of the reduced order model

An example on a two dof. model

▷ **Modal dynamics:**

$$\ddot{X}_1^2 + \omega_1^2 X_1 + g_{11}^1 X_1^2 + g_{22}^1 X_2^2 + g_{12}^1 X_1 X_2 + h_{111}^1 X_1^3 + h_{122}^1 X_1 X_2^2 = 0,$$
$$\ddot{X}_2^2 + \omega_2^2 X_2 + g_{22}^2 X_2^2 + g_{11}^2 X_1^2 + g_{12}^2 X_1 X_2 + h_{222}^2 X_2^3 + h_{112}^2 X_1^2 X_2 = 0.$$

An example on a two dof. model

▷ **Modal dynamics:**

$$\ddot{X}_1^2 + \omega_1^2 X_1 + g_{11}^1 X_1^2 + g_{22}^1 X_2^2 + g_{12}^1 X_1 X_2 + h_{111}^1 X_1^3 + h_{122}^1 X_1 X_2^2 = 0,$$
$$\ddot{X}_2^2 + \omega_2^2 X_2 + g_{22}^2 X_2^2 + g_{11}^2 X_1^2 + g_{12}^2 X_1 X_2 + h_{222}^2 X_2^3 + h_{112}^2 X_1^2 X_2 = 0.$$

▷ **Nonlinear change of variables:**

$$(X_p, Y_p = \dot{X}_p) \longrightarrow (R_p, S_p = \dot{S}_p)$$

$$\begin{pmatrix} X_p \\ Y_p = \dot{X}_p \end{pmatrix} = \begin{pmatrix} R_p \\ S_p = \dot{R}_p \end{pmatrix} + \begin{pmatrix} \mathcal{P}_p^{(3)}(R_i, S_i) \\ \mathcal{Q}_p^{(3)}(R_i, S_i) \end{pmatrix}$$

An example on a two dof. model

▷ **Modal dynamics:**

$$\ddot{X}_1^2 + \omega_1^2 X_1 + g_{11}^1 X_1^2 + g_{22}^1 X_2^2 + g_{12}^1 X_1 X_2 + h_{111}^1 X_1^3 + h_{122}^1 X_1 X_2^2 = 0,$$
$$\ddot{X}_2^2 + \omega_2^2 X_2 + g_{22}^2 X_2^2 + g_{11}^2 X_1^2 + g_{12}^2 X_1 X_2 + h_{222}^2 X_2^3 + h_{112}^2 X_1^2 X_2 = 0.$$

▷ **Nonlinear change of variables:**

$$(X_p, Y_p = \dot{X}_p) \longrightarrow (R_p, S_p = \dot{S}_p)$$

$$\begin{pmatrix} X_p \\ Y_p = \dot{X}_p \end{pmatrix} = \begin{pmatrix} R_p \\ S_p = \dot{R}_p \end{pmatrix} + \begin{pmatrix} \mathcal{P}_p^{(3)}(R_i, S_i) \\ \mathcal{Q}_p^{(3)}(R_i, S_i) \end{pmatrix}$$

▷ **Normal form:**

$$\ddot{R}_1^2 + \omega_1^2 R_1 + \left(A_{111}^1 + h_{111}^1\right) R_1^3 + B_{111}^1 R_1 \dot{R}_1^2 + R_1 \tilde{\mathcal{P}}_1^{(2)}(R_i, \dot{R}_i) + \dot{R}_1 \tilde{\mathcal{Q}}_1^{(2)}(R_i, \dot{R}_i) = 0$$
$$\ddot{R}_2^2 + \omega_1^2 R_2 + \left(A_{222}^2 + h_{222}^2\right) R_2^3 + B_{222}^2 R_2 \dot{R}_2^2 + R_2 \tilde{\mathcal{P}}_2^{(2)}(R_i, \dot{R}_i) + \dot{R}_2 \tilde{\mathcal{Q}}_1^{(2)}(R_i, \dot{R}_i) = 0$$

An example on a two dof. model

▷ **Modal dynamics:**

$$\ddot{X}_1^2 + \omega_1^2 X_1 + g_{11}^1 X_1^2 + g_{22}^1 X_2^2 + g_{12}^1 X_1 X_2 + h_{111}^1 X_1^3 + h_{122}^1 X_1 X_2^2 = 0,$$
$$\ddot{X}_2^2 + \omega_2^2 X_2 + g_{22}^2 X_2^2 + g_{11}^2 X_1^2 + g_{12}^2 X_1 X_2 + h_{222}^2 X_2^3 + h_{112}^2 X_1^2 X_2 = 0.$$

▷ **Nonlinear change of variables:**

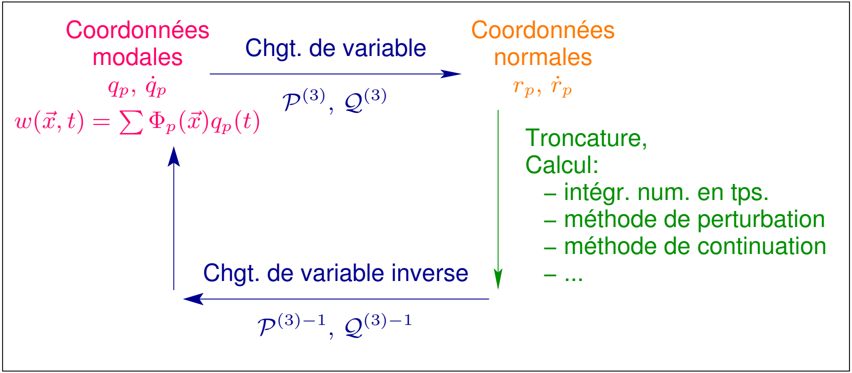
$$(X_p, Y_p = \dot{X}_p) \longrightarrow (R_p, S_p = \dot{S}_p)$$
$$\begin{pmatrix} X_p \\ Y_p = \dot{X}_p \end{pmatrix} = \begin{pmatrix} R_p \\ S_p = \dot{R}_p \end{pmatrix} + \begin{pmatrix} \mathcal{P}_p^{(3)}(R_i, S_i) \\ \mathcal{Q}_p^{(3)}(R_i, S_i) \end{pmatrix}$$

▷ **Invariant oscillators:** $R_2 = \dot{R}_2 = 0$

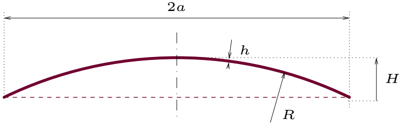
$$\ddot{R}_1^2 + \omega_1^2 R_1 + \left(A_{111}^1 + h_{111}^1\right) R_1^3 + B_{111}^1 R_1 \dot{R}_1^2 + \cancel{R_1 \tilde{\mathcal{P}}_1^{(2)}(R_i, \dot{R}_i) + \dot{R}_1 \tilde{\mathcal{Q}}_1^{(2)}(R_i, \dot{R}_i)} = 0$$
$$\cancel{\ddot{R}_2^2 + \omega_1^2 R_2 + \left(A_{222}^2 + h_{222}^2\right) R_2^3 + B_{222}^2 R_2 \dot{R}_2^2 + R_2 \tilde{\mathcal{P}}_2^{(2)}(R_i, \dot{R}_i) + \dot{R}_2 \tilde{\mathcal{Q}}_1^{(2)}(R_i, \dot{R}_i)} = 0$$

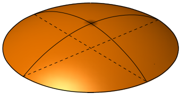
- all non resonant terms have been canceled: only the resonant cubic terms remains
- the dynamics can be exactly truncated to only one cubic oscillator ⇔ one nonlinear mode

An overview of normal form ROM



Hardening / softening behaviour of a spherical shell



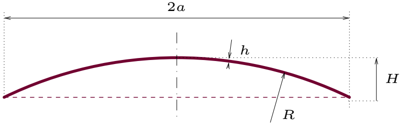


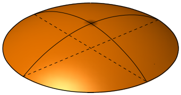
▷ Modal model

$$\underset{\{1,\dots,K\}}{\forall p} \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{K}\sum_{j\geq i}^{K} g_{ij}^p X_i X_j + \sum_{i=1}^{K}\sum_{j\geq i}^{K}\sum_{k\geq j}^{K} h_{ijk}^p X_i X_j X_k = 0.$$

$X_p(t)$: modal amplitude of the p-th. mode

Hardening / softening behaviour of a spherical shell





▷ Modal model

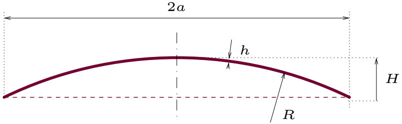
$$\underset{\{1,\dots,K\}}{\forall p} \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{K}\sum_{j\geq i}^{K} g_{ij}^p X_i X_j + \sum_{i=1}^{K}\sum_{j\geq i}^{K}\sum_{k\geq j}^{K} h_{ijk}^p X_i X_j X_k = 0.$$

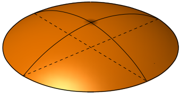
$X_p(t)$: modal amplitude of the p-th. mode

▷ Truncated dynamics

- One "linear" mode $\rightsquigarrow$ $\ddot{X}_p^2 + \omega_p^2 X_p + g_{pp}^p X_p^2 + h_{ppp}^p X_p^3 = 0$
- One nonlinear mode: $\rightsquigarrow$ $\ddot{R}_p^2 + \omega_p^2 R_p + \left(A_{ppp}^p + h_{ppp}^p\right) R_p^3 + B_{ppp}^p R_p \dot{R}_p^2 = 0$

Hardening / softening behaviour of a spherical shell





▷ Modal model

$$\underset{\{1,\dots,K\}}{\forall p} \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{K}\sum_{j\geq i}^{K} g_{ij}^p X_i X_j + \sum_{i=1}^{K}\sum_{j\geq i}^{K}\sum_{k\geq j}^{K} h_{ijk}^p X_i X_j X_k = 0.$$

$X_p(t)$: modal amplitude of the p-th. mode

▷ Truncated dynamics

- One "linear" mode $\rightsquigarrow$ $\ddot{X}_p^2 + \omega_p^2 X_p + g_{pp}^p X_p^2 + h_{ppp}^p X_p^3 = 0$
- One nonlinear mode: $\rightsquigarrow$ $\ddot{R}_p^2 + \omega_p^2 R_p + \left(A_{ppp}^p + h_{ppp}^p\right) R_p^3 + B_{ppp}^p R_p \dot{R}_p^2 = 0$

▷ Free oscillations

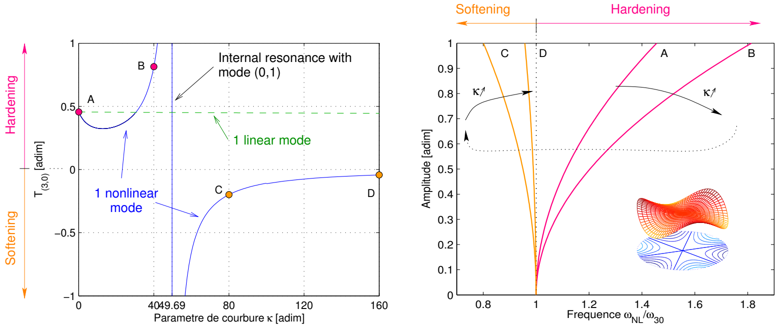
$$X_p, R_p = a\cos(\omega_{nl} + \varphi), \quad \omega_{nl} = \omega_p(1 + T a^2) \quad \rightsquigarrow \quad \text{sign of } T$$

- One "linear" mode $\rightsquigarrow$ T depends on g_{pp}^p only
- One nonlinear mode: $\rightsquigarrow$ T depends on A_{ppp}^p & B_{ppp}^p $\Rightarrow$ all the g_{ij}^p and the modes $i \neq p$.

Hardening / softening behaviour of a spherical shell

▷ Results

Mode (3,0) of a spherical shell as a function of curvature



$\kappa \propto 1/R$: curvature parameter

- the hardening/softening behaviour depends on the shell curvature: it is hardening for low curvatures ("plate" behaviour) and becomes softening for larger curvature.
- a "drastic" truncation to only one linear mode fails to predict the correct behaviour
  ⇝ importance of the other modes than the considered one, through the quadratic nonlinear terms

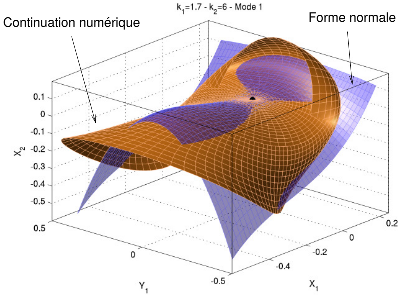
Validity range and performances

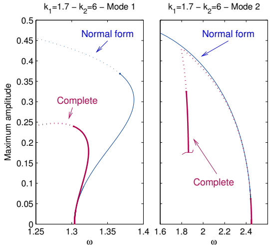
▷ **Vibrations libres**

⇝ Exact modal truncation,

⇝ Asymptotic expansion with a given validity range. . .

⇝ . . . difficult to predict *a priori* [Lamarque, Touzé, Thomas ND 2011]





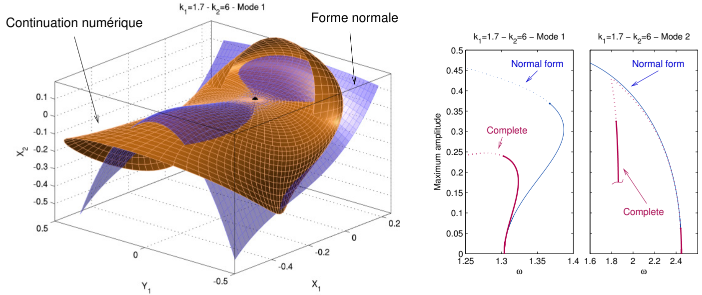
Validity range and performances

▷ **Vibrations libres**

⇝ Exact modal truncation,

⇝ Asymptotic expansion with a given validity range. . .

⇝ . . . difficult to predict *a priori* [Lamarque, Touzé, Thomas ND 2011]



▷ **Forced vibrations** [Touzé, Amabili, Thomas 2008]

⇝ Non-exact modal truncation,

⇝ good results in practice

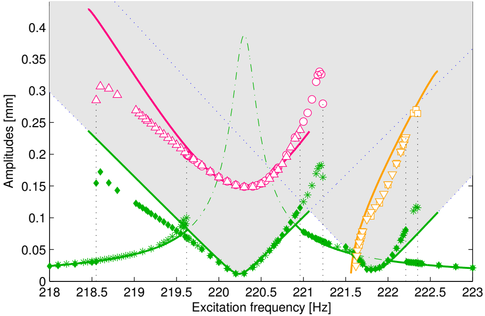
A priori writing of reduced order models

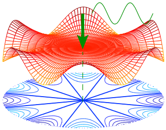
▷ **A 1:1:2 internal resonance in a spherical cap**

$\omega_3 \simeq 2\omega_1 \simeq 2\omega_2$

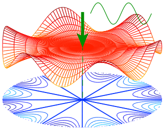
▷ **Normal form**

$$\begin{cases} \ddot{q}_1 + 2\xi_1\omega_1\dot{q}_1 + \omega_1^2 q_1 = \beta_1 q_1 q_3 \\ \ddot{q}_2 + 2\xi_2\omega_2\dot{q}_2 + \omega_2^2 q_2 = \beta_2 q_2 q_3 \\ \ddot{q}_3 + 2\xi_2\omega_2\dot{q}_3 + \omega_2^2 q_3 = \beta_3 q_1^2 + \beta_4 q_2^2 + Q\cos\Omega t \end{cases}$$

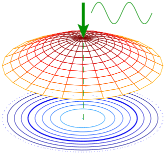




Mode 1 (6,0,cos) - ω_1



Mode 1 (6,0,sin) - ω_2

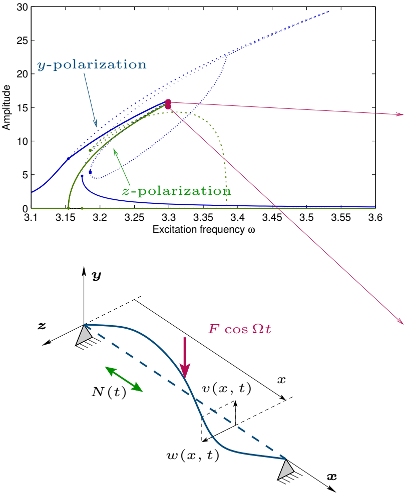


Mode 1 (0,1) - ω_3

[Thomas, Touzé, *et al.*, IJSS 2005, ND 2007]

A 1:1 internal resonance in a string

▷ **Coupling between y and z polarizations**



""

Conclusions and perspectives

▷ **a rapid overview of nonlinear modes**

▷ **give precise insights on both periodic ans quasiperiodic regimes**

⇝ they give the skeleton of the dynamics

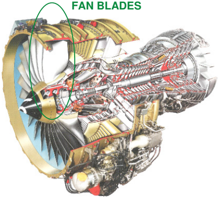
▷ **several definitions, several methods of computation**

- Periodic orbits: well suited for numerics and analysis, not for ROM. No damping.
- Invariant manifolds: analysis, ROM & damping, but difficult in numerics (internal resonances)
- Normal forms: analysis, ROM, damping, internal resonances, but impossible in numerics

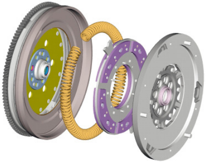
▷ **in the future ?**

⇝ using the power of numerical continuation methods to compute the invariant manifolds and use them to built reduced order models, even with internal resonances . . . ?

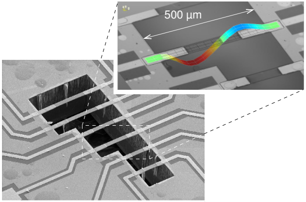
Thank you for your attention



Jetengine fan blades

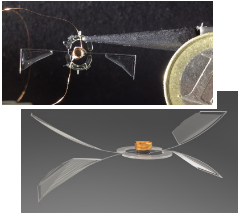


Vibration absorbers



mass micro sensors

Steel pans



Nano-drone