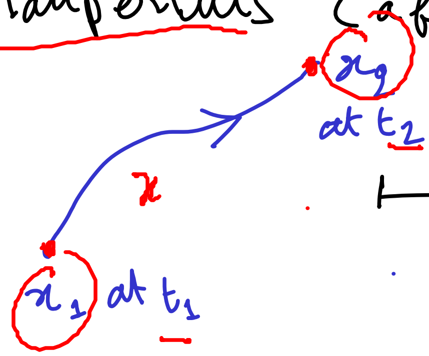
Multisymplectic geometry: some perspectives

Newton $\quad m \dfrac{d^2x}{dt^2} = -\nabla V(x) \qquad V(x) = -\dfrac{GMm}{|x|}$

Maupertuis (after Henon, Fermat ...)

$x = (x^1, x^2, x^3)$



$$\longmapsto \quad \mathcal{L}[x] = \int_{t_1}^{t_2} \left(m \frac{|\dot{x}|^2}{2} - V(x) \right) dt$$

... Euler, Lagrange ... Hamilton, Jacobi ....

$$\mathcal{L}[x] = \int_{t_1}^{t_2} L(x(t), \dot{x}(t))\, dt$$

Hamilton

$$\begin{cases} p_i = \dfrac{\partial L}{\partial \dot{x}^i}(x, \dot{x}) = m\, \dot{x}^i \\ H(x,p) = p_i \dot{x}^i - L(x, \dot{x}) = \underbrace{\dfrac{|p|^2}{2m}}_{E_c} + V(x) \end{cases}$$

EL.

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}} \longrightarrow \boxed{\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial x^i} \end{aligned}} \quad \text{Hamilton}$$

Gain? 1) Simplifies the changes of variables
$\longrightarrow$ helps to solve
2) Hamilton-Jacobi method

$$S(t,x): \quad \frac{\partial S}{\partial t} + H\left(x, \frac{\partial S}{\partial x^i}\right) = 0$$

Why? Because a geometrical description is possible.

$$\text{Space} = \{ (t, x^1, x^2, x^3, p_1, p_2, p_3) \} \quad (7 \text{ dimensional})$$

$$\text{Form} \quad \boxed{\theta = p_1\,dx^1 + p_2\,dx^2 + p_3\,dx^3 - H(t,x,p)\,dt}$$

Poincaré, Cartan ("energy-movement quantity tensor")

Geometrically: length of an infinitesimal displacement

$$\theta(\nearrow) = \text{an infinitesimal number}$$

Physically?

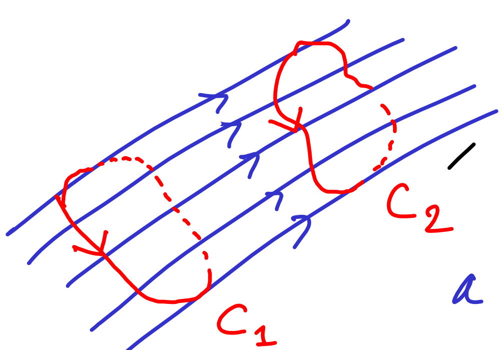
1) Along a trajectory γ

dans variété presymplectique

γ, t_1, t_2

$$\int_\gamma \theta = \text{the action} = \int_{t_1}^{t_2} L(x, \dot{x})\,dt.$$

2) Along a loop sliding along a tube of trajectories



$$\int_{C_1} \theta = \int_{C_2} \theta$$

a conserved quantity!

("invariant intégral")

Discovered and used by Poincaré

Cartan $\longrightarrow$ new principle

Conservation of the quantity of movement and of energy (Elie Cartan)

"If we agree that a trajectory is defined by a succession of states which constitutes a solution of a system of ordinary differential equations, this system is,

among all possible imaginable systems of differential equations

characterized by the property of admitting, as an integral invariant, the curvilinear integral on an arbitrary closed curve of the tensor of quantity of movement and energy."

$$\int_C \theta \qquad \text{where } \theta = p_i\, dx^i - H\,dt$$

Mathematical implementation → Exterior Differential Calculus

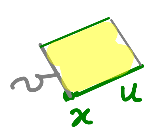
Objects are "forms"

0-form = function = at each point of the space it gives a number

1-form α = at each point, it helps to measure tangent vectors

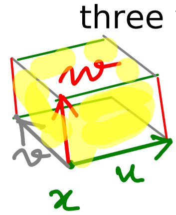
$$\longmapsto \quad \alpha_x(u)$$

2-form β = at each point, it helps to measure the area spanned by two vectors



$$\longmapsto \quad \beta_x(u, v)$$

3-form γ = at each point, it measures the volume spanned by three vectors



$$\longmapsto \quad \gamma_x(u, v, w)$$

etc.

Fundamental operations

- $\wedge$: exterior product
- $\lrcorner$: interior product
- d : exterior differential
- φ^* : pull-back operation.

The preceding principle reads :

For any vector u and if γ is a trajectory

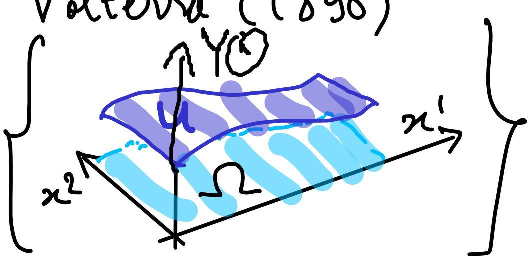
$$\boxed{u \lrcorner d\theta \big|_{\gamma} = 0 .}$$

Applications :
- to Mechanics (céleste) 1800
- to Quantum Physics 1920
- to "Completely Integrable Systems". 1970

What is the Multisymplectic formalism ?

The same theory for variational problems with several variables:

Volterra (1890)



$$\longmapsto \mathcal{L}[u] = \int_\Omega L(x, u(x), \nabla u(x)) dx$$

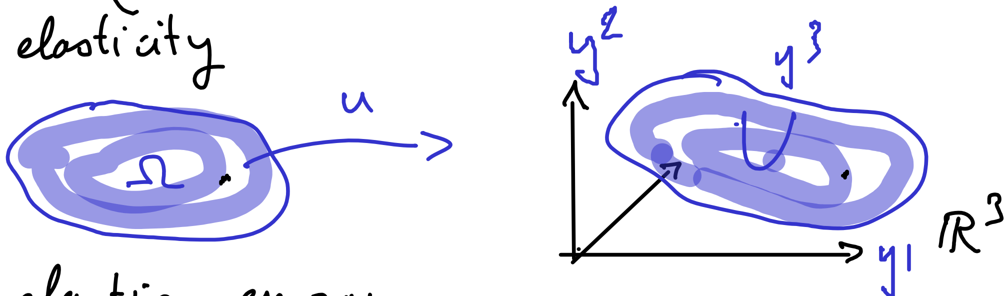
$$\begin{cases} p_i^\alpha = \dfrac{\partial L}{\partial \frac{\partial u^i}{\partial x^\alpha}} (x, u, \nabla u) \\ H(x,u,p) = \displaystyle\sum_{i,\alpha} p_i^\alpha \frac{\partial u^i}{\partial x^\alpha} - L(x, u, \nabla u) \end{cases}$$

E-L $\quad \dfrac{\partial}{\partial x^\alpha}\left(\dfrac{\partial L}{\partial \frac{\partial u^i}{\partial x^\alpha}}\right) = \dfrac{\partial L}{\partial u^i} \longmapsto$

$$\boxed{\begin{aligned} \frac{\partial u^i}{\partial x^\alpha} &= \frac{\partial H}{\partial p^\alpha_i} \\ \sum_\alpha \frac{\partial p^\alpha_i}{\partial x^\alpha} &= -\frac{\partial H}{\partial u^i} \end{aligned}}$$

De Donder-Weyl (1935)
Volterra (1890)

Example : elasticity



$\int_\Omega L(u, Du) = \mathcal{L}[u]$ = elastic energy.
The multisymplectic space $\{(x^\alpha, y^i, p^\alpha_i)\}$ has 15 dimensions.

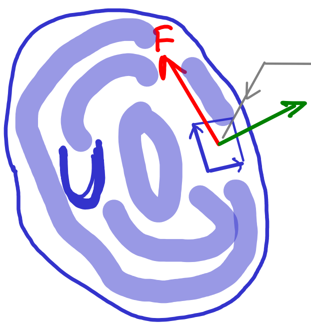
$$\boxed{\begin{aligned} \theta = \; & p^\alpha_i \, dy^i \wedge \Big(\frac{\partial}{\partial x^\alpha} \lrcorner \underbrace{dx^1 \wedge dx^2 \wedge dx^3}_{dx}\Big) \\ & - H \, dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}}$$

3-forme: $\dfrac{\partial}{\partial x^1} \lrcorner dx = dx^2 \wedge dx^3$

If Γ^3 is the "graph" of u in the multisymplectic space,

énergie élastique: $\displaystyle\int_\Gamma \theta = \mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$

Recover the Cauchy stress tensor :



infinitesimal area face $(da)^2$ on the boundary of U

n : unit normal vector to the infinitesimal face

$F = (F_1, F_2, F_3)$ = force on the body U through the infinitesimal face

$$F_1 = (\sigma_{11} n^1 + \sigma_{12} n^2 + \sigma_{13} n^3)\,(da)^2$$
$$F_2 = (\sigma_{21} n^1 + \sigma_{22} n^2 + \sigma_{23} n^3)\,(da)^2, \text{ etc}$$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \text{Cauchy stress tensor}$$

By a virtual deformation governed by a vector field X the energy of the body increases of:

$$\delta W = -\int_{\partial U} \langle F(n), X \rangle \, d\sigma$$

If ξ lifts X in the multisymplectic space, this variation of energy is equal to:

$$\delta W = \int_U L_\xi \theta = \int_U 0 + d(\xi \lrcorner \theta) = \int_{\partial U} \xi \lrcorner \theta$$

Actually

$$\xi \lrcorner \theta \Big|_{\text{face}(n)} = -\langle F(n), X \rangle \, d\sigma$$

Case where ξ (or X) is a symmetry: then

$$0 = \int_U L_\xi \theta = \boxed{\int_{\partial U} \xi \lrcorner \theta}$$

Conservation law (cf. Archimedes' principle)

<u>More generally</u>: <u>Noether first theorem</u>

to: each symmetry of a variational problems it corresponds: a conservation law satisfied by any solution.

symmetry $\simeq$ vector field ξ such that

$$\underbrace{L_\xi \theta}_{=0} = d\varphi$$

pour une symétrie

Cartan's (magic) formula : $L_\xi \theta = \xi \lrcorner d\theta + d(\xi \lrcorner \theta)$

$$\Rightarrow \quad d(\xi \lrcorner \theta - \varphi) = \xi \lrcorner d\theta$$

By restricting on a solution Γ of the Hamilton-Volterra equations:

$$d(\xi \lrcorner \theta - \varphi)|_\Gamma = \xi \lrcorner d\theta|_\Gamma = 0$$

Hence if U is contained in Γ and has a boundary

$$\boxed{\int_{\partial U} (\xi \lrcorner \theta - \varphi) = 0.}$$

Example : ξ is a rectilinear translation in space

and for elasticity, $\varphi = 0$, hence

$$\int_{\partial U} \xi \lrcorner \theta = 0$$

It says that the projection along ξ of the forces on U through ∂U vanishes.

More general problems: dynamical problems with several variables (waves)

- same theory
- the physical meaning of the action is different.

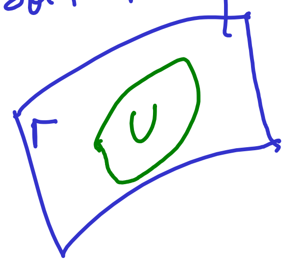
Perspectives

- Many developments towards Quantum Fields Theory
- Recent work on Yang-Mills (H.) and work in progress on gravitation

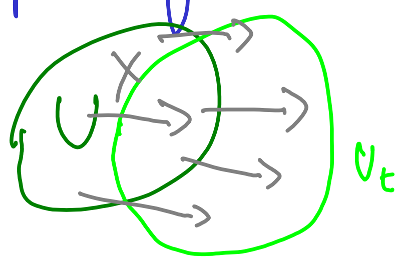
ouvert
- Still few applications of the Hamilton-Jacobi theory
- Theory of integrable systems to be developped

Compléments

Soit $\Gamma \subset \{$variété multisymplectique$\}$ et U un ouvert $\subset \Gamma$.



Soit X un champ de vecteur et U_t l'image par le flot de X au bout du temps t.



On note $\Phi_t = \exp tX$ si bien que : $U_t = \Phi_t(U)$.

$$\int_{U_t} \theta = \int_{\Phi_t(U)} \theta = \int_U \Phi_t^* \theta$$

$$\Rightarrow \frac{d}{dt} \int_{U_t} \theta \Big|_{t=0} = \int_U \frac{d}{dt} \left(\Phi_t^* \theta \right) \Big|_{t=0} = \int_U L_X \theta$$

$$= \int_U \underbrace{X \lrcorner d\theta}_{} + d(X \lrcorner \theta) = 0 + \int_{\partial U} X \lrcorner \theta$$

l'intégrale de cette forme sur U s'annule car $U \subset \Gamma$ et Γ représente une solution des équations de Hamilton-Volterra

Conclusion : on obtient l'identité :

$$\boxed{\frac{d}{dt}\int_{U_t}\theta = \int_{\partial U} X \lrcorner \theta}$$, valable si

$U \subset T$: <u>solution des équations de Hamilton-Volterra.</u>

→ Si en plus X est une symétrie, alors $\int_{U_t}\theta$ ne dépend pas de t et donc le membre de gauche s'annule → loi de conservation.

<u>Exemple</u> : pour l'équation $\left\| \begin{array}{l} \Box\varphi + m^2\varphi = 0 \\ \left(\Box = \frac{\partial^2}{\partial t^2} - \Delta\right). \end{array}\right.$ (Klein-Gordon) sur $\mathbb{R}^4$

$\mathcal{M}$ = variété multisymplectique
$= \{(x^0, x^1, x^2, x^3, \varphi, e, p^0, p^1, p^2, p^3)\}$

muni de la 4-forme $\theta = e\,dx + p^\mu d\varphi \wedge dx_\mu$

où $\begin{cases} dx = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ dx_\mu = \frac{\partial}{\partial x^\mu} \lrcorner dx \end{cases}$

L'Hamiltonien est : $\mathcal{H} = e + \frac{1}{2}\eta_{\mu\nu}p^\mu p^\nu + \frac{1}{2}m^2\varphi^2$

où $(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ (métrique de Minkowski sur l'espace-temps).

a) Si $X = \frac{\partial}{\partial x^\mu}$ (translation) $\quad P_\mu := \frac{\partial}{\partial x^\mu} \lrcorner \theta$

Le relèvement de X dans $\mathcal{M}$ est l'unique champ de vecteurs ξ tel que :

$$dP_\mu + \xi_\mu \lrcorner d\theta = 0$$

Or $0 = L_X \theta = d(X \lrcorner \theta) + X \lrcorner d\theta = d P_\mu + X \lrcorner d\theta$

$\Rightarrow$ par unicité de ξ_μ,

$$\xi_\mu = X_\mu = \frac{\partial}{\partial x^\mu}.$$

Les équations de la dynamique s'obtiennent en restreignant θ (et $d\theta$) à l'hypersurface $\mathcal{H} = 0 \Leftrightarrow e = -\frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu - \frac{m^2\varphi^2}{2}$

Une solution est représentée par une sous-variété Γ de dimension 4, contenue dans $\{\mathcal{H}=0\}$ et satisfaisant

$$\boxed{\forall Y \text{ tel que } d\mathcal{H}(Y) = 0, \quad Y \lrcorner d\theta\big|_\Gamma = 0}$$

Ici

$$\Gamma = \{ (x, \varphi(x), e(x), p(x)) \, ; \, x \in \mathbb{R}^4 \}$$

avec $p^\mu(x) = \eta^{\mu\nu} \frac{\partial \varphi}{\partial x^\nu}(x)$ et $e(x)$ tel que $\mathcal{H}(x, \varphi(x), e(x), p(x)) = 0$

Alors $$\int_{\{x^0=0\}\cap\Gamma} \theta = \int_{\{x^0=0\}\cap\Gamma} e \, dx_0 - p^\nu d\varphi \wedge dx_{\nu 0}$$

$$= \int_{\mathbb{R}^3} \left(e + p^1 \frac{\partial \varphi}{\partial x^1} + p^2 \frac{\partial \varphi}{\partial x^2} + p^3 \frac{\partial \varphi}{\partial x^3} \right) \Big|_{x^0=0} dx_0$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} \left[\left(\frac{\partial \varphi}{\partial x^0}\right)^2 + \left(\frac{\partial \varphi}{\partial x^1}\right)^2 + \left(\frac{\partial \varphi}{\partial x^2}\right)^2 + \left(\frac{\partial \varphi}{\partial x^3}\right)^2 \right] dx_0$$

$$= - \text{énergie}.$$

b) Si $X = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu}$ (générateur infinitésimal du groupe de Lorentz : rotations et "boosts").

on obtient $\Phi^\mu_\nu := X \lrcorner \theta = x^\mu \Phi_\nu - x^\nu \Phi_\mu$

Le champ de vecteur ξ^μ_ν qui relève Φ^μ_ν est l'unique solution de :

$$d\Phi^\mu_\nu + \xi^\mu_\nu \lrcorner d\theta = 0$$

On trouve $\xi^\mu_\nu = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu} + p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu}$.