Multisymplectic Lie group variational integrators
Part 2: application to a geometrically exact beam in $\mathbb{R}^3$

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***Abstract*—The focus of this paper is to study and test a Lie group multisymplectic integrator (Part 1) for the particular case of a geometrically exact beam. We exploit the multisymplectic character of the integrator to analyze the energy and momentum map conservations associated to the temporal and spatial discrete evolutions. This allows us to explore the temporal motion of the beam and the spatial evolution of the wave motion through the beam.**

***Index Terms*—multisymplectic structure, discrete mechanics, variational integrator, Lie group symmetry, discrete global Noether theorem, geometrically exact beam model.**

I. Introduction

In [2] and [4], the geometrically exact model for elastic beams has been developed from the point of view of classical mechanics. In this paper, we employ the field theoretic covariant description of geometrically exact beams, presented in Part I [1]. An important feature of the proposed multisymplectic point of view is that it allows not only the description of the behavior of the beam during an interval of time, a classical dynamics point of view, but also the description of the evolution in space of the deformations of the beam when the "time" evolution of the strain at a boundary node is known.

II. Covariant formulation of the geometrically exact beam

The configuration of the geometrically exact beam is described by its line of centroids, modeled by a map

$$\mathbf{r} : s \in [0, L] \mapsto \mathbf{r}(s) \in \mathbb{R}^3,$$

and the orientation of all its cross-sections at points $\mathbf{r}(s)$, given by a moving orthonormal basis $\{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$ satisfying $\mathbf{d}_I(s) = \Lambda(s)\mathbf{E}_I,\ I = 1, 2, 3$, where the moving basis is described by a map

$$\Lambda : s \in [0, L] \to \Lambda(s) \in SO(3)$$

where $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a fixed orthonormal basis.

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In the covariant formulation, the configuration variables are the spacetime maps (or fields)

$$g : X \ni (s, t) \mapsto g(s, t) = (\Lambda(t, s), \mathbf{r}(t, s)) \in G,$$

where $X := [0, T] \times [0, L]$ and $G = SE(3)$. Here, $SE(3)$ denotes the special Euclidean group of orientation preserving rotations and translations and $\mathfrak{se}(3)$ is its Lie algebra. The fields g have to be interpreted as sections of the (here trivial) fiber bundle $\pi : X \times G \to X$, in contrast to the dynamic formulation where the configuration variables are curves $t \mapsto (\Lambda(t), \mathbf{r}(t))$ in the infinite dimensional space $\mathcal{F}(\mathcal{B}, G)$ of maps from $\mathcal{B} := [0, L]$ to G.

The Lagrangian density depends on $(g, \dot{g}, g')$, where $g := (\Lambda, \mathbf{r})$, $\dot{} := \partial_t$ and $' := \partial_s$. We define the convective velocities and strains $\xi, \eta \in \mathfrak{se}(3)$ by

$$\begin{aligned} \xi &:= (\omega, \gamma) := g^{-1}\dot{g} = (\Lambda^{-1}\dot{\Lambda}, \Lambda^{-1}\dot{\mathbf{r}}), \\ \eta &:= (\Omega, \mathbf{\Gamma}) := g^{-1}g' = (\Lambda^{-1}\Lambda', \Lambda^{-1}\mathbf{r}'). \end{aligned} \tag{1}$$

The trivialization map $(g, \xi, \eta) \mapsto (g, \dot{g}, g')$ induces a trivialized version of the Lagrangian density, which can be written as

$$\begin{aligned} \mathcal{L}(\Lambda, \mathbf{r}, \boldsymbol{\omega}, \boldsymbol{\gamma}, \mathbf{\Omega}, \mathbf{\Gamma}) &= \frac{1}{2}\langle \mathbb{J}\xi, \xi\rangle - \frac{1}{2}\langle \mathbb{C}\ (\eta - \mathbf{E}_6), (\eta - \mathbf{E}_6)\rangle - \Pi(g) \\ &=: K(\xi) - \Phi(\eta) - \Pi(g) = \mathcal{L}(g, \xi, \eta), \end{aligned}$$

where $\mathbf{E}_6 = (0, 0, 0, 0, 0, 1) \in \mathbb{R}^6$. Considering that the thickness of the rod is constant, small compared to its length, and that the material is homogeneous and isotropic, the matrices $\mathbb{J}$ and $\mathbb{C}$ are given by

$$\mathbb{J} = \begin{bmatrix} J & 0 \\ 0 & M\mathbf{I}_3 \end{bmatrix}, \qquad \mathbb{C} = \begin{bmatrix} \mathbf{C}_2 & 0 \\ 0 & \mathbf{C}_1 \end{bmatrix}.$$

Here M is the total mass of the beam and J its inertia tensor, both assumed to be constant. The diagonal matrices $\mathbf{C}_1, \mathbf{C}_2$ are given in [3]. Recall that K, Φ, and Π correspond, respectively, to the kinetic energy density, the bending energy density, and the potential energy density. The bold face letters are the images of light faced letters under the standard isomorphism $\mathfrak{se}(3) \cong \mathbb{R}^6$ given by $\mathfrak{se}(3) \ni (\omega, \gamma) \longmapsto (\boldsymbol{\omega}, \boldsymbol{\gamma}) \in \mathbb{R}^6$, where $\omega = \widehat{\boldsymbol{\omega}} \in \mathfrak{so}(3)$ is defined by $\widehat{\boldsymbol{\omega}}\mathbf{v} := \boldsymbol{\omega} \times \mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^3$, and $\gamma = \boldsymbol{\gamma} \in \mathbb{R}^3$.

The action functional is obtained by spacetime integration of the Lagrangian density $\mathcal{L}$, i.e.,

$$\mathfrak{S}(g(\cdot)) = \int_X \mathcal{L}(g, \xi, \eta).$$

The Covariant Hamilton Principle $\delta\mathcal{S} = 0$, for variations δg vanishing at the boundary, yields the trivialized covariant

Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial K}{\partial \xi} - \mathrm{ad}^*_\xi \frac{\partial K}{\partial \xi} = \frac{d}{ds}\frac{\partial \Phi}{\partial \eta} - \mathrm{ad}^*_\eta \frac{\partial \Phi}{\partial \eta} - g^{-1}\frac{\partial \Pi}{\partial g}, \tag{2}$$

where $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual map to $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_\xi\, \eta := [\xi, \eta]$.

III. Covariant variational integrator

In the covariant formulation, the time and space variables are treated in the same way and we take advantage of this fact to formulate the geometric discretization of the beam. The spacetime discretization is realized by fixing a time step Δt and a space step Δs, and decomposing the interval $[0, T]$ into N subintervals $[t^j, t^{j+1}]$, $j \in \{0, ..., N-1\}$, and the interval $[0, L]$ into A subintervals $[s_a, s_{a+1}]$, $a \in \{0, ..., A-1\}$. In order to fit with the left trivialization, the discretization of the spacetime domain is based on a triangular decomposition (Figure 1), where a triangle $\triangle^j_a$ is has vertices

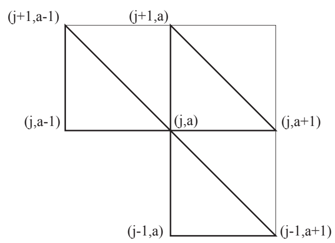

Figure 1. The triangles $\triangle^j_a, \triangle^{j-1}_a, \triangle^j_{a-1}$.

$$((j, a), (j+1, a), (j, a+1)),\ \ j = 0, ..., N-1,\ \ a = 0, ..., A-1.$$

The set of all triangles $\triangle$ in spacetime $[0, T] \times [0, L]$ is denoted $X^\triangle_d$. The action functional associated to the Lagrangian density (2) is approximated on the square $\square^j_a$, with vertices

$$((j, a), (j+1, a), (j, a+1), (j+1, a+1)),$$

by the discrete Lagrangian $\mathfrak{L}_d : X^\triangle_d \times G \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given for the beam by

$$\begin{aligned}\mathfrak{L}_d(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a) &= \Delta t \Delta s K(\xi^j_a) - \Delta t \Delta s \left[\Phi(\eta^j_a) + \Pi(g^j_a)\right] \\ &= \frac{1}{2}\langle \mathbb{J}\xi^j_a, \xi^j_a\rangle - \frac{1}{2}\langle \mathbb{C}\,(\eta^j_a - \mathbf{E}_6), (\eta^j_a - \mathbf{E}_6)\rangle - \Pi(g^j_a).\end{aligned} \tag{3}$$

Then, as indicated in Part I [1], the Discrete Covariant Euler-Lagrange (DCEL) equations are

$$\begin{aligned}&\frac{1}{\Delta t}\left(-\mu^j_a + \mathrm{Ad}^*_{\tau(\Delta t \xi^{j-1}_a)}\mu^{j-1}_a\right) + \frac{1}{\Delta s}\left(\lambda^j_a - \mathrm{Ad}^*_{\tau(\Delta s \eta^j_{a-1})}\lambda^j_{a-1}\right) \\ &\quad - (g^j_a)^{-1} D_{g^j_a}\Pi(g^j_a) = 0,\end{aligned} \tag{4}$$

for all $j = 1, ..., N-1$, and $a = 1, ..., A-1$. Where

$$\mu^j_a := \left(\mathrm{d}\tau^{-1}_{\Delta t \xi^j_a}\right)^* \partial_\xi K\left(\xi^j_a\right), \quad \lambda^j_a := \left(\mathrm{d}\tau^{-1}_{\Delta s \eta^j_a}\right)^* \partial_\eta \Phi\left(\eta^j_a\right),$$

The complete algorithm obtained via the covariant variational integrator (4) can be implemented, through time or space, according to the following steps:

Time Integrator

Given: $g^j_a, \xi^{j-1}_a, \mu^{j-1}_a, f^j_a,$ **for** $a = 0, ..., A,$

Compute:

$$\eta^j_a = \frac{1}{\Delta s}\tau^{-1}\left((g^j_a)^{-1} g^j_{a+1}\right),$$

$$\lambda^j_a = \left(\mathrm{d}\,\tau^{-1}_{\Delta s \eta^j_a}\right)^* \partial_\eta \Phi\left(\eta^j_a\right),$$

$$\mu^j_a = \mathrm{Ad}^*_{\tau(\Delta t \xi^{j-1}_a)}\,\mu^{j-1}_a + \begin{cases} \Delta t\left(\frac{1}{\Delta s}\lambda^j_a - (g^j_a)^{-1} D_g\Pi(g^j_a)\right), & \textbf{for } a = 0, \\ \Delta t\left(\frac{1}{\Delta s}\left(\lambda^j_a - \mathrm{Ad}^*_{\tau(\Delta s \eta^j_{a-1})}\,\lambda^j_{a-1}\right) - (g^j_a)^{-1} D_g\Pi(g^j_a)\right), & \textbf{for } a = 1, ..., A-1, \end{cases}$$

Solve the discrete Legendre transform: $\mu^j_a = \left(\mathrm{d}\,\tau^{-1}_{\Delta t \xi^j_a}\right)^* \partial_\xi K\left(\xi^j_a\right)$, **for** ξ^j_a

Update: $g^{j+1}_a = g^j_a \tau\left(\Delta t \xi^j_a\right).$

Space Integrator

Given: $g^j_a, \eta^{j-1}_a, \lambda^{j-1}_a, D_g\Pi(g^j_a),$ **for** $j = 0, ..., N,$

Compute:

$$\xi^j_a = \frac{1}{\Delta t}\tau^{-1}\left((g^j_a)^{-1} g^{j+1}_a\right),$$

$$\mu^j_a := \left(\mathrm{d}^R\,\tau^{-1}_{\Delta t \xi^j_a}\right)^* \partial_\xi K\left(\xi^j_a\right),$$

$$\lambda^j_a = \mathrm{Ad}^*_{\tau(\Delta s \eta^j_{a-1})}\,\lambda^j_{a-1} - \begin{cases} \Delta s\left(\frac{1}{\Delta t}\mu^j_a + (g^j_a)^{-1} D_g\Pi(g^j_a)\right), & \textbf{for } j = 0, \\ \Delta s\left(\frac{1}{\Delta t}\left(\mu^j_a - \mathrm{Ad}^*_{\tau(\Delta t \xi^{j-1}_a)}\,\mu^{j-1}_a\right) + (g^j_a)^{-1} D_g\Pi(g^j_a)\right), & \textbf{for } j = 1, ..., N-1, \end{cases}$$

Solve the discrete Legendre transform: $\lambda^j_a = -\left(\mathrm{d}\,\tau^{-1}_{\Delta s \eta^j_a}\right)^* \partial_\eta \Phi\left(\eta^j_a\right)$, **for** η^j_a

Update: $g^j_{a+1} = g^j_a \tau\left(\Delta s\, \eta^j_a\right).$

Figure 2. Algorithms in time and in space.

a) Discrete Legendre transforms.: In terms of the discrete Lagrangian $\mathfrak{L}_d(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a)$ in (3), the corresponding discrete Legendre transforms

$$\mathbb{F}^k \mathfrak{L}_d : X^\triangle_d \times G \times \mathfrak{g} \times \mathfrak{g} \to G \times \mathfrak{g}^*, \quad k = 1, 2, 3,$$

are

$$\begin{aligned}&\mathbb{F}^1\mathfrak{L}_d(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a) \\ &\quad = \left(g^j_a, -\Delta s \mu^j_a + \Delta t \lambda^j_a - \Delta t \Delta s (g^j_a)^{-1} D_{g^j_a}\Pi_d(g^j_a)\right), \\ &\mathbb{F}^2\mathfrak{L}_d(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a) \\ &\quad = \left(g^{j+1}_a, \Delta s \mathrm{Ad}^*_{\tau(\Delta t \xi^j_a)}\mu^j_a\right), \\ &\mathbb{F}^3\mathfrak{L}_d(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a) \\ &\quad = \left(g^j_{a+1}, -\Delta t \mathrm{Ad}^*_{\tau(\Delta s \eta^j_a)}\lambda^j_a\right).\end{aligned}$$

b) Discrete covariant momentum maps.: Given is the symmetry group $H \subseteq G$ acting on G by multiplication on the left. The infinitesimal generator of the left multiplication by H on G, associated to the Lie algebra element $\zeta \in \mathfrak{h}$, is expressed as $\zeta_G(g) = \zeta g$. The three discrete Lagrangian momentum maps

$$J^k_{\mathfrak{L}_d} : X^\triangle_d \times G \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}^*, \quad k = 1, 2, 3,$$

are defined by

$$\begin{aligned}&\left\langle J^k_{\mathfrak{L}_d}(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a), \zeta\right\rangle \\ &\quad = \left\langle \mathbb{F}^k\mathfrak{L}_d(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a),\ (g(k))^{-1}\zeta_G(g(k))\right\rangle, \quad \zeta \in \mathfrak{h},\end{aligned}$$

where $g(1) = g^j_a$, $g(2) = g^{j+1}_a$, and $g(3) = g^j_{a+1}$. For the discrete Lagrangian $\mathcal{L}_d$ defined in (3), the discrete momentum maps, abbreviated $J^k_{\mathcal{L}_d}(\triangle^j_a)$, are

$$
\begin{aligned}
J^1_{\mathcal{L}_d}(\triangle^j_a) &= i^*\mathrm{Ad}^*_{(g^j_a)^{-1}}\left(-\Delta s\mu^j_a + \Delta t\lambda^j_a - \Delta t\Delta s(g^j_a)^{-1}\mathbf{D}_{g^j_a}\Pi_d(g^j_a)\right),\\
J^2_{\mathcal{L}_d}(\triangle^j_a) &= i^*\,\mathrm{Ad}^*_{(g^{j+1}_a)^{-1}}\left(\Delta s\mathrm{Ad}^*_{\tau(\Delta t\xi^j_a)}\mu^j_a\right),\\
J^3_{\mathcal{L}_d}(\triangle^j_a) &= i^*\,\mathrm{Ad}^*_{(g^j_{a+1})^{-1}}\left(-\Delta t\mathrm{Ad}^*_{\tau(\Delta s\eta^j_a)}\lambda^j_a\right),
\end{aligned}
$$

where $i^* : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ is the dual map to the Lie algebra inclusion $i : \mathfrak{h} \rightarrow \mathfrak{g}$.

Assuming H-invariance of the discrete covariant Lagrangian $\mathcal{L}_d$ and assuming that the discrete Euler-Lagrange equations are satisfied, we get the discrete global Noether theorem:

$$
\begin{aligned}
&\sum_{j=1}^{N-1}\left(J^1_{\mathcal{L}_d}(\triangle^j_0) + J^2_{\mathcal{L}_d}(\triangle^{j-1}_0) + J^3_{\mathcal{L}_d}(\triangle^j_{A-1})\right)\\
&+\sum_{a=1}^{A-1}\left(J^1_{\mathcal{L}_d}(\triangle^0_a) + J^2_{\mathcal{L}_d}(\triangle^{N-1}_a) + J^3_{\mathcal{L}_d}(\triangle^0_{a-1})\right)\\
&+J^1_{\mathcal{L}_d}(\triangle^0_0) + J^2_{\mathcal{L}_d}(\triangle^{N-1}_0) + J^3_{\mathcal{L}_d}(\triangle^0_{A-1}) = 0.
\end{aligned}
\tag{5}
$$

c) Symplectic properties of the time and space discrete evolutions.: The discrete time-evolution Lagrangian $\mathsf{L}_d : M^{A+1} \times M^{A+1} \rightarrow \mathbb{R}$, and space-evolution Lagrangian $\mathsf{N}_d : M^{N+1} \times M^{N+1} \rightarrow \mathbb{R}$, are given in Part I [1].

Given an H-invariant Lagrangian $\mathcal{L}_d$, both L_d and N_d inherit this H-invariance. The discrete Lagrangian momentum maps $\mathbf{J}^{\pm}_{\mathsf{L}_d} : SE(3)^A \times \mathfrak{se}(3)^A \rightarrow \mathfrak{h}^*$ and $\mathbf{J}^{\pm}_{\mathsf{N}_d} : SE(3)^N \times \mathfrak{se}(3)^N \rightarrow \mathfrak{h}^*$ are given by

$$
\begin{aligned}
\mathbf{J}^-_{\mathsf{L}_d}(\mathbf{g}^j, \boldsymbol{\xi}^j) &= -\sum_{a=0}^{A-1}\left(J^1_{\mathcal{L}_d}(\triangle^j_a) + J^3_{\mathcal{L}_d}(\triangle^j_a)\right),\\
\mathbf{J}^+_{\mathsf{L}_d}(\mathbf{g}^j, \boldsymbol{\xi}^j) &= \sum_{a=0}^{A-1} J^2_{\mathcal{L}_d}(\triangle^j_a),\\
\mathbf{J}^-_{\mathsf{N}_d}(\mathbf{g}_a, \boldsymbol{\eta}_a) &= -\sum_{j=0}^{N-1}\left(J^1_{\mathcal{L}_d}(\triangle^j_a) + J^2_{\mathcal{L}_d}(\triangle^j_a)\right),\\
\mathbf{J}^+_{\mathsf{N}_d}(\mathbf{g}_a, \boldsymbol{\eta}_a) &= \sum_{j=0}^{N-1} J^3_{\mathcal{L}_d}(\triangle^j_a).
\end{aligned}
$$

If the configuration is prescribed at the temporal extremities and zero-traction boundary conditions are used, then the discrete momentum map $\mathbf{J}_{\mathsf{L}_d} = \mathbf{J}^+_{\mathsf{L}_d} = \mathbf{J}^-_{\mathsf{L}_d}$ is conserved. In general, conservation of $\mathbf{J}^{\pm}_{\mathsf{N}_d}$ does not hold in this case.

If the configuration is prescribed at the spatial extremities and zero-momentum boundary conditions are used, then the discrete momentum map $\mathbf{J}_{\mathsf{N}_d} = \mathbf{J}^+_{\mathsf{N}_d} = \mathbf{J}^-_{\mathsf{N}_d}$ is conserved. In general, conservation of $\mathbf{J}^{\pm}_{\mathsf{L}_d}$ does not hold in this case.

IV. Test: space-integration and time-reconstruction

In this example the mesh is defined by the space step $\Delta s = 0.05$ and the time step $\Delta t = 0.2$. The total length of the beam is $L = 1\,m$ and the total simulation time is $T = 30\,s$. The characteristics of the material are: $\rho = 10^3\,kg/m^3$, $M = 10^{-1}\,kg/m$, $E = 5.10^4\,N/m^2$, $\nu = 0.35$.

d) Space-integration.: We assume that we know the evolution (for all $t \in [0, T]$) of one of the extremities, $a = 0$, as well as the evolution of its strain (for all $t \in [0, T]$), i.e.

$$
\begin{aligned}
&g^0_0 = (\mathrm{Id}, (0,0,0)), \quad g^{j+1}_0 = g^j_0\,\tau(\Delta t\xi^j_0), \quad \text{for all } j \neq 0,\\
&\quad \text{where } \xi^j_0 = (0, -2, 0, 0, -0.1, 0),\\
&g^0_1 = (\mathrm{Id}, (0,0,\Delta s)), \quad g^{j+1}_1 = g^j_1\,\tau(\Delta t\xi^j_1), \quad \text{for all } j \neq 0,\\
&\quad \text{with } \xi^j_1 = (0.007, -1.998, -0.007, -0.08, -0.1, 0).
\end{aligned}
$$

We also assume that the velocity of the beam is zero, at the initial and final times $t = 0, T$. i.e. the configuration is prescribed at the spatial extremities and zero-momentum boundary conditions are used.

This example, corresponds to the rotation of a beam around an axis combined with a displacement like an air-screw

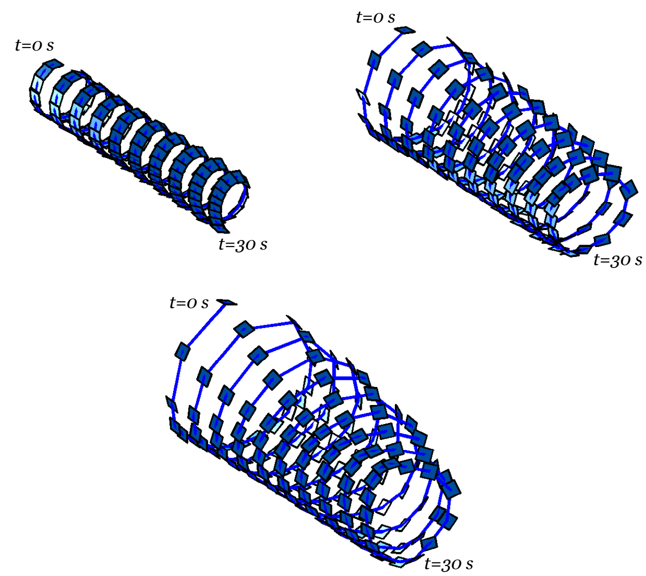

Figure 3. Each figure represents the time evolution $\mathbf{g}_a = \{g^j_a, j = 1, ..., N-1\}$ of a given node a of the beam, with $t^{N-1} = 30s$. The chosen nodes correspond to $s = 0.35\,m$, $0.75\,m$, $1\,m$.

We checked numerically that the discrete global Noether theorem (5) is verified. Moreover we observe that the discrete Noether theorems for $\mathbf{J}^{\pm}_{\mathsf{N}_d}$ is verified as we observe in Fig. 4.

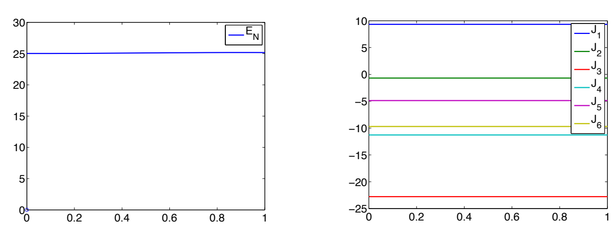

Figure 4. Left: total “energy” behavior $\mathsf{E}_{\mathsf{N}_d}$. Right: conservation of the discrete momentum map $\mathbf{J}_{\mathsf{N}_d} = (\mathsf{J}^1, ..., \mathsf{J}^6) \in \mathbb{R}^6$. Both, during a space interval of 1m.

e) Time-reconstruction.: The resulting motion is depicted in Fig. 5.

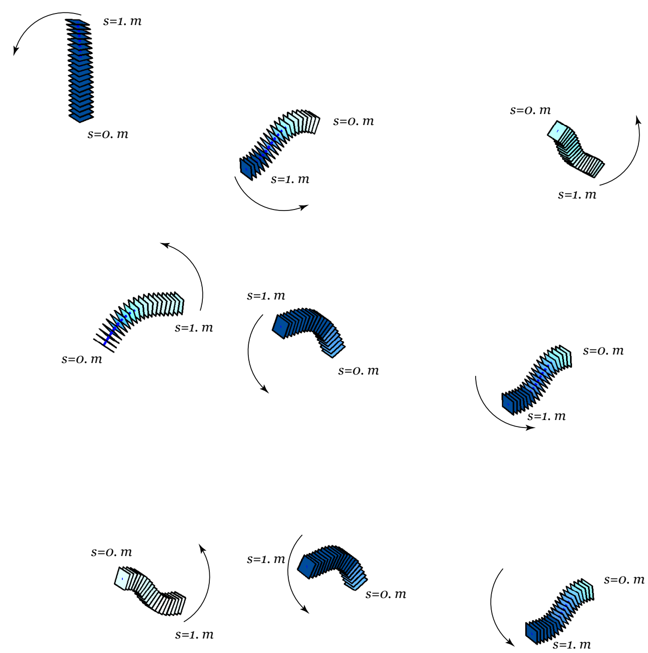

Figure 5. From left to right and top to bottom: time-reconstruction of the beam sections at times $t = 0.2\,s$, $0.8\,s$, $2.2\,s$, $3.0\,s$, $3.6\,s$, $4.2\,s$, $5.6\,s$, $6.8\,s$, $7.4\,s$.

The behavior of the discrete energy $\mathsf{E}_{\mathsf{L}_d}$ and momentum maps $\mathbf{J}^{\pm}_{\mathsf{L}_d}$ is illustrated in Fig. 6.

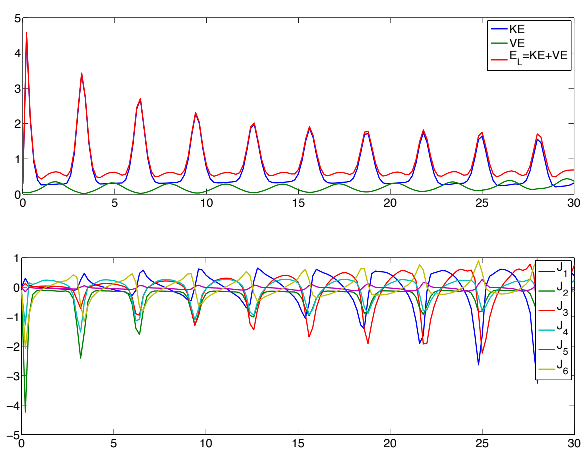

Figure 6. Left: total energy behavior $\mathsf{E}_{\mathsf{L}_d}$. Right: momentum map behavior $\mathbf{J}_{\mathsf{L}_d} = (\mathsf{J}^1, ..., \mathsf{J}^6)$. Both, during a time interval of 30s, where we observe periodicity due to rotations.

We note that the discrete energy $\mathsf{E}_{\mathsf{L}_d}$ and momentum maps $\mathbf{J}^{\pm}_{\mathsf{L}_d}$ associated to the temporal evolution is not conserved for this problem. This is due to the fixed boundary conditions $\mathbf{g}^j_0$, $\mathbf{g}^j_1$, $j = 0, ..., N$, in $a = 0, 1$.

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